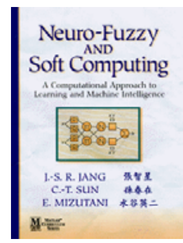
Chapter 3: Fuzzy Rules & Fuzzy Reasoning

- Extension Principle & Fuzzy Relations (3.2)
- Fuzzy if-then Rules(3.3)
- Fuzzy Reasonning (3.4)



Jyh-Shing Roger Jang et al., *Neuro-Fuzzy and Soft Computing: A Computational Approach to Learning and Machine Intelligence*, First Edition, Prentice Hall, 1997

Extension Principle & Fuzzy Relations (3.2)

- Extension principle

A is a fuzzy set on *X* :

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_n)/x_n$$

The image of *A* under *f(.)* is a fuzzy set *B:*

$$B = \mu_B(x_1)/y_1 + \mu_B(x_2)/y_2 + \cdots + \mu_B(x_n)/y_n$$

where $y_i = f(x_i)$, $i = 1$ to n

If *f(.)* is a many-to-one mapping, then

$$\mu_B(y) = \max_{x = f^{-1}(y)} \mu_A(x)$$

Extension Principle & Fuzzy Relations (3.2) (cont.)

– Example:

Application of the extension principle to fuzzy sets with discrete universes

Let A = 0.1 / -2+0.4 / -1+0.8 / 0+0.9 / 1+0.3 / 2
and f(x) = x2 – 3

Applying the extension principle, we obtain:
B = 0.1 / 1+0.4 / -2+0.8 / -3+0.9 / -2+0.3 /1
= 0.8 / -3+(0.4V0.9) / -2+(0.1V0.3) / 1
= 0.8 / -3+0.9 / -2+0.3 / 1

where "V" represents the "max" operator

Same reasoning for continuous universes

Extension Principle & Fuzzy Relations (3.2) (cont.)

- Fuzzy relations

– A fuzzy relation R is a 2D MF:

$$R = \{((x, y), \mu_R(x, y)) | (x, y) \in X \times Y\}$$

– Examples:

Let X = Y = IR+
and R(x,y) = "y is much greater than x"
The MF of this fuzzy relation can be subjectively defined as:

if X = {

$$\mu_R(x,y) = \begin{cases} \frac{y-x}{x+y+2}, & \text{if } y > x \\ 0, & \text{if } y \le x \end{cases}$$

Extension Principle & Fuzzy Relations (3.2) (cont.)

- Then R can be Written as a matrix:

$$\mathbf{R} = \begin{bmatrix} \mathbf{0} & \mathbf{0.111} & \mathbf{0.200} & \mathbf{0.273} & \mathbf{0.333} \\ \mathbf{0} & \mathbf{0} & \mathbf{0.091} & \mathbf{0.167} & \mathbf{0.231} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0.077} & \mathbf{0.143} \end{bmatrix}$$

where R{i,j} = μ[xi, yj]

  - x is close to y (x and y are numbers)
  - x depends on y (x and y are events)
  - x and y look alike (x and y are persons or objects)
  - If x is large, then y is small (x is an observed reading and Y is a corresponding action)

Extension Principle & Fuzzy Relations (3.2) (cont.)

– Max-Min Composition

- The max-min composition of two fuzzy relations R_1 (defined on X and Y) and R_2 (defined on Y and Z) is

$$\mu_{R_1 \circ R_2}(x, z) = \vee_y [\mu_{R_1}(x, y) \wedge \mu_{R_2}(y, z)]$$

- Properties:
  - Associativity: $R \circ (S \circ T) = (R \circ S) \circ T$
  - Distributivity over union: $R \circ (S \bigcup T) = (R \circ S) \bigcup (R \circ T)$
  - Week distributivity over intersection:

$$R \circ (S \bigcap T) \subseteq (R \circ S) \bigcap (R \circ T)$$

  - Monotonicity: $S \subseteq T \Rightarrow (R \circ S) \subseteq (R \circ T)$

Extension Principle & Fuzzy Relations (3.2) (cont.)

- Max-min composition is not mathematically tractable, therefore other compositions such as max-product composition have been suggested

– Max-product composition

$$\mu_{R_1 \circ R_2}(x, z) = \vee_y [\mu_{R_1}(x, y)\mu_{R_2}(y, z)]$$

Extension Principle & Fuzzy Relations (3.2) (cont.)

– Example of max-min & max-product composition

- Let R_1 = "x is relevant to y"
  R_2 = "y is relevant to z"
  be two fuzzy relations defined on X*Y and Y*Z respectively X = {1,2,3}, Y = {α,β,χ,δ} and Z = {a,b}.

  Assume that:

$$\mathbf{R_1} = \begin{bmatrix} \mathbf{0.1} & \mathbf{0.3} & \mathbf{0.5} & \mathbf{0.7} \\ \mathbf{0.4} & \mathbf{0.2} & \mathbf{0.8} & \mathbf{0.9} \\ \mathbf{0.6} & \mathbf{0.8} & \mathbf{0.3} & \mathbf{0.2} \end{bmatrix} \qquad \mathbf{R_2} = \begin{bmatrix} \mathbf{0.9} & \mathbf{0.1} \\ \mathbf{0.2} & \mathbf{0.3} \\ \mathbf{0.5} & \mathbf{0.6} \\ \mathbf{0.7} & \mathbf{0.2} \end{bmatrix}$$

Extension Principle & Fuzzy Relations (3.2) (cont.)

The derived fuzzy relation "x is relevant to z" based on R_1 & R_2

Let's assume that we want to compute the degree of relevance between $2 \in X$ & $a \in Z$

Using max-min, we obtain:

$$\mu_{R_1 \circ R_2}(2,a) = \max\{0.4 \wedge 0.9, 0.2 \wedge 0.2, 0.8 \wedge 0.5, 0.9 \wedge 0.7\}$$
$$= \max\{0.4, 0.2, 0.5, 0.7\}$$
$$= 0.7$$

Using max-product composition, we obtain:

$$\mu_{R_1 \circ R_2}(2,a) = \max\{0.4 * 0.9, 0.2 * 0.2, 0.8 * 0.5, 0.9 * 0.7\}$$
$$= \max\{0.36, 0.04, 0.40, 0.63\}$$
$$= 0.63$$

Fuzzy if-then rules (3.3)

- Linguistic Variables
  - Conventional techniques for system analysis are intrinsically unsuited for dealing with systems based on human judgment, perception & emotion
  - Principle of incompatibility
    - As the complexity of a system increases, our ability to make precise & yet significant statements about its behavior decreases until a fixed threshold
    - Beyond this threshold, precision & significance become almost mutually exclusive characteristics [Zadeh, 1973]

Fuzzy if-then rules (3.3) (cont.)

– The concept of linguistic variables introduced by Zadeh is an alternative approach to modeling human thinking

– Information is expressed in terms of fuzzy sets instead of crisp numbers

– **Definition:** A linguistic variable is a quintuple (x, T(x), X, G, M) where:

- x is the name of the variable
- T(x) is the set of linguistic values (or terms)
- X is the universe of discourse
- G is a syntactic rule that generates the linguistic values
- M is a semantic rule which provides meanings for the linguistic values

Fuzzy if-then rules (3.3) (cont.)

– Example:

A numerical variable takes numerical values

Age = 65

A linguistic variables takes linguistic values

Age is old

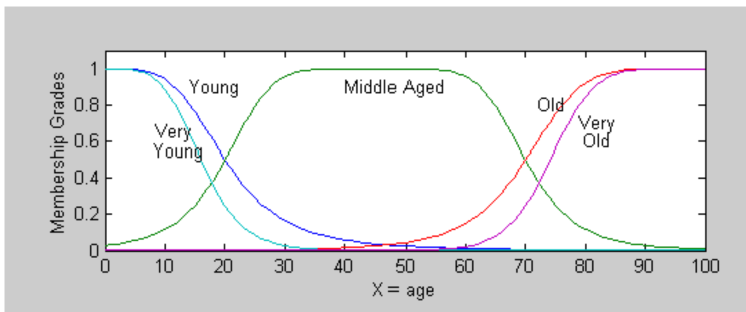
A linguistic value is a fuzzy set

All linguistic values form a term set

T(age) = {young, not young, very young, ...
middle aged, not middle aged, ...
old, not old, very old, more or less old, ...
not very yound and not very old, ...}

Fuzzy if-then rules (3.3) (cont.)

- Where each term T(age) is characterized by a fuzzy set of a universe of discourse X= = [0,100]



Fuzzy if-then rules (3.3) (cont.)

- The syntactic rule refers to the way the terms in T(age) are generated
- The semantic rule defines the membership function of each linguistic value of the term set
- The term set consists of primary terms as (young, middle aged, old) modified by the negation ("not") and/or the hedges (very, more or less, quite, extremely,…) and linked by connectives such as (and, or, either, neither,…)

Fuzzy if-then rules (3.3) (cont.)

– Concentration & dilation of linguistic values

- Let A be a linguistic value described by a fuzzy set with membership function $\mu_A(.)$

  – $$A^k = \int_X [\mu_A(x)]^k / x$$

  is a modified version of the original linguistic value.

  – A^2 = CON(A) is called the concentration operation

  – $\sqrt{A}$ = DIL(A) is called the dilation operation

  – CON(A) & DIL(A) are useful in expression the hedges such as "very" & "more or less" in the linguistic term A

  – Other definitions for linguistic hedges are also possible

Fuzzy if-then rules (3.3) (cont.)

– Composite linguistic terms

Let's define:

$$\text{NOT}(A) = \neg A = \int_X [1 - \mu_A(x)] / x,$$

$$A \text{ and } B = A \cap B = \int_X [\mu_A(x) \wedge \mu_B(x)] / x$$

$$A \text{ or } B = A \cup B = \int_X [\mu_A(x) \vee \mu_B(x)] / x$$

where A, B are two linguistic values whose semantics are respectively defined by $\mu_A(.)$ & $\mu_B(.)$

Composite linguistic terms such as: "not very young", "not very old" & "young but not too young" can be easily characterized

Fuzzy if-then rules (3.3) (cont.)

– Example: Construction of MFs for composite linguistic terms

Let's $\mu_{young}(x) = bell(x,20,2,0) = \dfrac{1}{1+\left(\dfrac{x}{20}\right)^4}$

$$\mu_{old}(x) = bell(x,30,3,100) = \frac{1}{1+\left(\dfrac{x-100}{30}\right)^6}$$

Where x is the age of a person in the universe of discourse [0, 100]

- More or less = DIL(old) = √old = $\displaystyle\int_X \sqrt{\frac{1}{1+\left(\dfrac{x-100}{30}\right)^6}} \Big/ x$

Fuzzy if-then rules (3.3) (cont.)

- Not young and not old = ¬young ∩ ¬old =

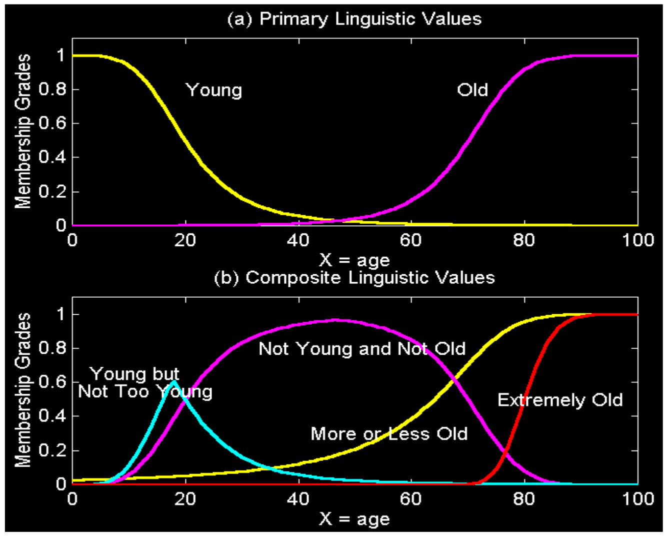
$$\int_X \left[1-\frac{1}{1+\left(\dfrac{x}{20}\right)^4}\right] \wedge \left[1-\frac{1}{1+\left(\dfrac{x-100}{30}\right)^6}\right] \Big/ x$$

- Young but not too young = young ∩ ¬young² (too = very) =

$$\int_x \left[\frac{1}{1+\left(\dfrac{x}{20}\right)^4}\right] \wedge \left[1-\left(\frac{1}{1+\left(\dfrac{x}{20}\right)^4}\right)^2\right] \Big/ x$$

- Extremely old ≡ very very very old = CON (CON(CON(old))) =

$$\int_x \left[\frac{1}{1+\left(\dfrac{x-100}{30}\right)^6}\right]^8 \Big/ x$$



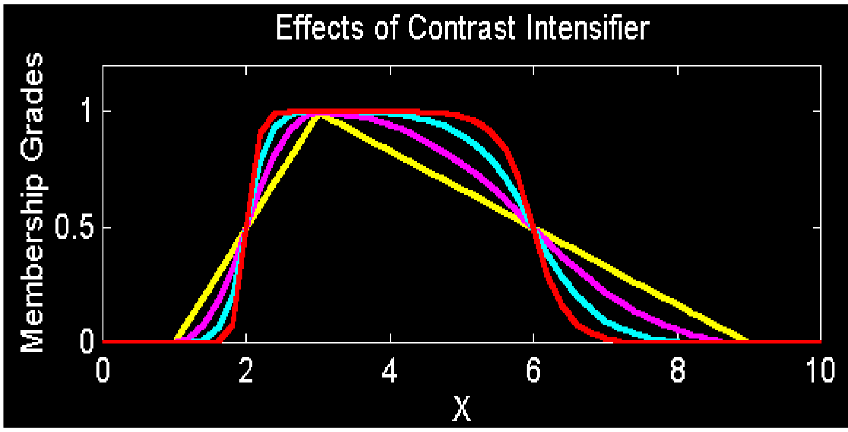
Fuzzy if-then rules (3.3) (cont.)

– Contrast intensification

the operation of contrast intensification on a linguistic value A is defined by

$$\mathrm{INT}(A) = \begin{cases} 2A^2 & \text{if } 0 \le \mu_A(x) \le 0.5 \\ \neg 2(\neg A)^2 & \text{if } 0.5 \le \mu_A(x) \le 1 \end{cases}$$

- INT increases the values of $\mu_A(x)$ which are greater than 0.5 & decreases those which are less or equal that 0.5
- Contrast intensification has effect of reducing the fuzziness of the linguistic value A



Fuzzy if-then rules (3.3) (cont.)

– Orthogonality

A term set T = $t_1, \ldots, t_n$ of a linguistic variable x on the universe X is orthogonal if:

$$\sum_{i=1}^{n} \mu_{t_i}(x) = 1, \quad \forall x \in X$$

Where the t_i's are convex & normal fuzzy sets defined on X.

Fuzzy if-then rules (3.3) (cont.)

- General format:
  - If x is A then y is B (where A & B are linguistic values defined by fuzzy sets on universes of discourse X & Y).
    - "x is A" is called the antecedent or premise
    - "y is B" is called the consequence or conclusion
  - Examples:
    - If pressure is high, then volume is small.
    - If the road is slippery, then driving is dangerous.
    - If a tomato is red, then it is ripe.
    - If the speed is high, then apply the brake a little.

Fuzzy if-then rules (3.3) (cont.)

- Meaning of fuzzy if-then-rules ($A \Rightarrow B$)
  - It is a relation between two variables x & y; therefore it is a binary fuzzy relation R defined on X * Y
  - There are two ways to interpret $A \Rightarrow B$:
    - A coupled with B
    - A entails B

    if A is coupled with B then:

$$\mathbf{R = A \Rightarrow B = A * B = \int_{X*Y} \mu_A(x) \tilde{*} \mu_B(y) / (x, y)}$$

where $\tilde{*}$ is a T - normoperator.

Fuzzy if-then rules (3.3) (cont.)

If A entails B then:

R = A ⇒ B = ¬ A ∪ B (material implication)

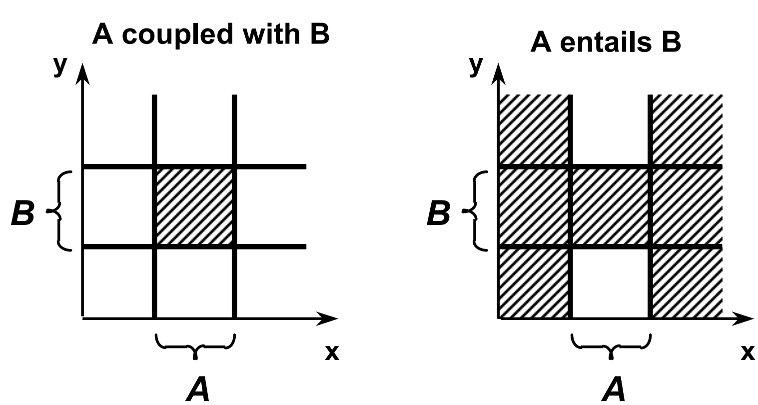
R = A ⇒ B = ¬ A ∪ (A ∩ B) (propositional calculus)

R = A ⇒ B = (¬ A ∩ ¬ B) ∪ B (extended propositional calculus)

$$\mu_R(x,y) = \sup\left\{c; \mu_A(x)\tilde{*}c \le \mu_B(y), 0 \le c \le 1\right\}$$

Fuzzy if-then rules (3.3) (cont.)

Two ways to interpret "If x is A then y is B":



Fuzzy if-then rules (3.3) (cont.)

- Note that R can be viewed as a fuzzy set with a two-dimensional MF

$$\mu_R(x, y) = f(\mu_A(x), \mu_B(y)) = f(a, b)$$

With $a = \mu_A(x)$, $b = \mu_B(y)$ and f called the fuzzy implication function provides the membership value of (x, y)

Fuzzy if-then rules (3.3) (cont.)

– Case of "A coupled with B"

$$R_m = A * B = \int_{X*Y} \mu_A(x) \wedge \mu_B(y)/(x, y)$$

(minimum operator proposed by Mamdani, 1975)

$$\mathbf{R_p = A * B = \int_{X*Y} \mu_A(x)\,\mu_B(y)/(x, y)}$$

(product proposed by Larsen, 1980)

$$\mathbf{R_{bp} = A * B = \int_{X*Y} \mu_A(x) \otimes \mu_B(y)/(x, y)}$$

$$\mathbf{= \int_{X*Y} 0 \vee (\mu_A(x) + \mu_B(y) - 1)/(x, y)}$$

(bounded product operator)

Fuzzy if-then rules (3.3) (cont.)

– Case of “A coupled with B” (cont.)

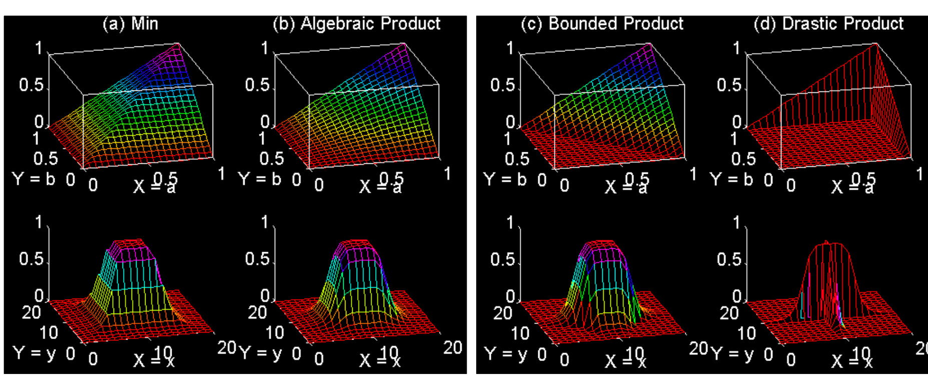
$$R_{dp} = A * B = \int_{X*Y} \mu_A(x) \hat{\cdot} \mu_B(y)/(x,y)$$

$$where: \mathrm{f(a,b)} = a \hat{\cdot} b = \begin{cases} a & \text{if b} = 1 \\ b & \text{if a} = 1 \\ 0 & \text{if otherwise} \end{cases}$$

$Example$ for $\mu_A(x) = bell(x;4,3,10)$ and $\mu_B(y) = bell(y;4,3,10)$

(Drastic operator)

Fuzzy if-then rules (3.3) (cont.)



A coupled with B

Fuzzy if-then rules (3.3) (cont.)

– Case of "A entails B"

$$R_a = \neg A \cup B = \int_{X*Y} 1 \wedge (1 - \mu_A(x) + \mu_B(y)/(x,y)$$

$$\text{where: } f_a(a,b) = 1 \wedge (1 - a + b)$$

(Zadeh's arithmetic rule by using bounded sum operator for union)

$$R_{mm} = \neg A \cup (A \cap B) = \int_{X*Y} (1 - \mu_A(x)) \vee (\mu_A(x) \wedge \mu_B(y))/(x,y)$$

$$\text{where: } f_m(a,b) = (1 - a) \vee (a \wedge b)$$

(Zadeh's max-min rule)

Fuzzy if-then rules (3.3) (cont.)

– Case of "A entails B" (cont.)

$$R_s = \neg A \cup B = \int_{X*Y} (1 - \mu_A(x)) \vee \mu_B(y)/(x,y)$$

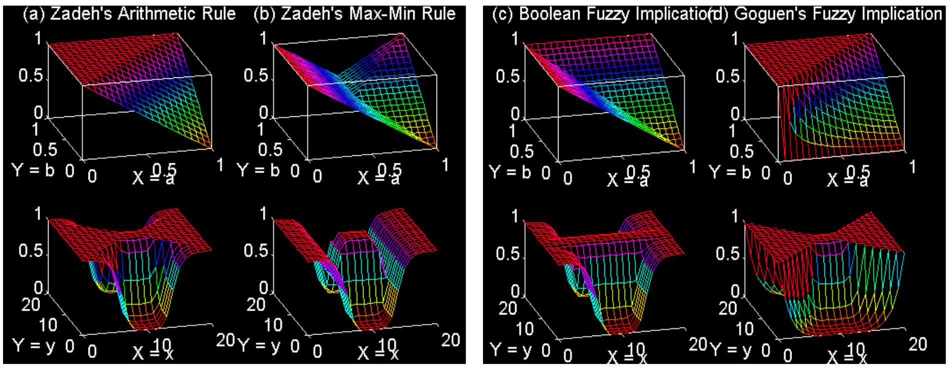
$$\text{where: } f_s(a,b) = (1 - a) \vee b$$

(Boolean fuzzy implication with max for union)

$$R_\Delta = \int_{X*Y} (\mu_A(x) \mathrel{\tilde{<}} \mu_B(y))/(x,y)$$

$$\text{where: } a \mathrel{\tilde{<}} b = \begin{cases} 1 & \text{if } a \le b \\ b/a & \text{otherwise} \end{cases}$$

(Goguen's fuzzy implication with algebraic product for T-norm)



A entails B

Fuzzy Reasoning (3.4)

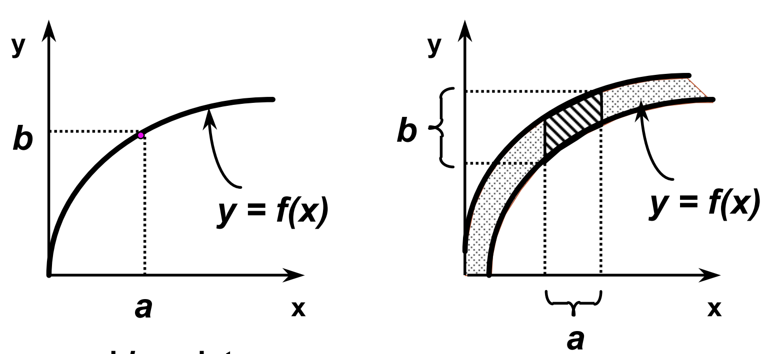
- Definition
  - Known also as approximate reasoning
  - It is an inference procedure that derives conclusions from a set of fuzzy if-then-rules & known facts

Fuzzy Reasoning (3.4) (cont.)

- Compositional rule of inference
  - Idea of composition (cylindrical extension & projection)
    - Computation of b given a & f is the goal of the composition
      - Image of a point is a point
      - Image of an interval is an interval

Fuzzy Reasoning (3.4) (cont.)

Derivation of $y = b$ from $x = a$ and $y = f(x)$:



***a* and *b*: points**
***y = f(x)* : a curve**

***a* and *b*: intervals**
***y = f(x)* : an interval-valued function**

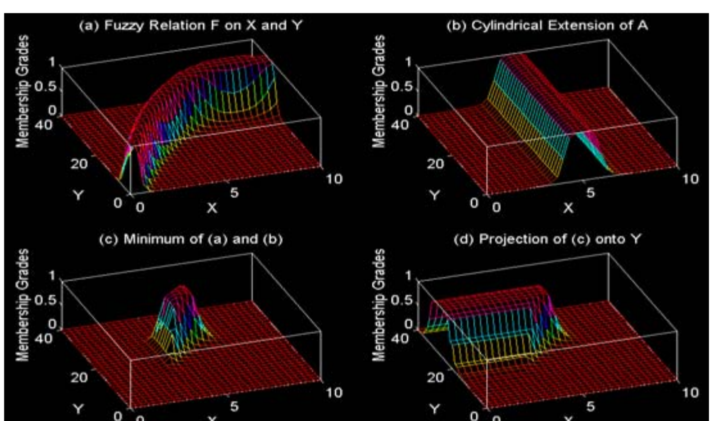
Fuzzy Reasoning (3.4) (cont.)

– The extension principle is a special case of the compositional rule of inference

- F is a fuzzy relation on X*Y, A is a fuzzy set of X & the goal is to determine the resulting fuzzy set B
  - Construct a cylindrical extension c(A) with base A
  - Determine c(A) $\wedge$ F (using minimum operator)
  - Project c(A) $\wedge$ F onto the y-axis which provides B

Fuzzy Reasoning (3.4) (cont.)

a is a fuzzy set and *y* = *f(x)* is a fuzzy relation:



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Fuzzy Reasoning (3.4) (cont.)

- Given A, A ⇒ B, infer B

A = “today is sunny”

A ⇒ B: day = sunny then sky = blue
infer: “sky is blue”

- illustration

Premise 1 (fact):	x is A
Premise 2 (rule):	if x is A then y is B
Consequence:	y is B

Fuzzy Reasoning (3.4) (cont.)

- Approximation

A’ = “ today is more or less sunny”

B’ = “ sky is more or less blue”

- illustration

Premise 1 (fact):	x is A’
Premise 2 (rule):	if x is A then y is B
Consequence:	y is B’

(approximate reasoning or fuzzy reasoning!)

Fuzzy Reasoning (3.4) (cont.)

- Definition of fuzzy reasoning

  Let A, A' and B be fuzzy sets of X, X, and Y, respectively. Assume that the fuzzy implication $A \Rightarrow B$ is expressed as a fuzzy relation R on X*Y. Then the fuzzy set B induced by "x is A' " and the fuzzy rule "if x is A then y is B' is defined by:

$$\mu_{B'}(y) = \max_{x} \min\left[\mu_{A'}(x), \mu_{R}(x,y)\right]$$

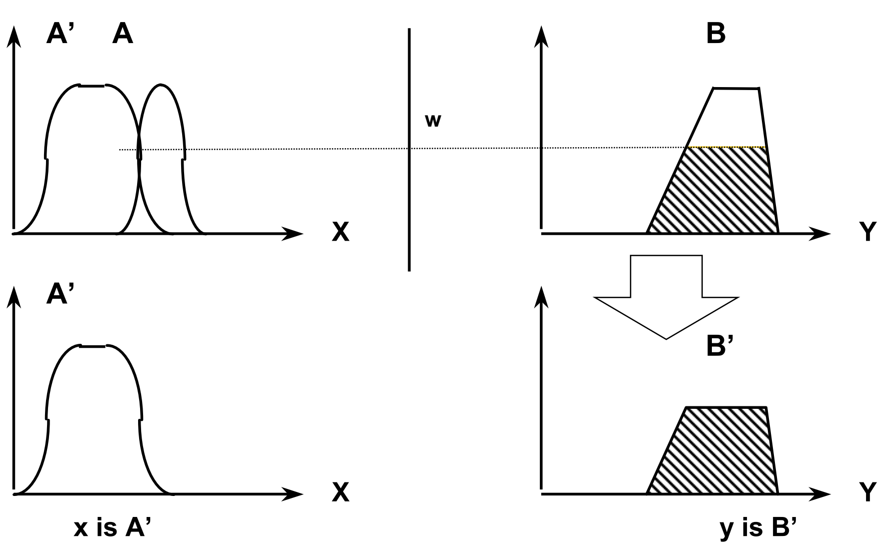
- Single rule with single antecedent

  Rule : if x is A then y is B

  Fact: x is A'

  Conclusion: y is B' ($\mu_{B'}(y) = [\vee_x (\mu_{A'}(x) \wedge \mu_A(x)] \wedge \mu_B(y)$)

Fuzzy Reasoning (3.4) (cont.)



– Single rule with multiple antecedents

Premise 1 (fact): x is A' and y is B'

Premise 2 (rule): if x is A and y is B then z is C

Conclusion: z is C'

Premise 2: A*B → C

$$R_{mamdani}(A,B,C) = (A*B)*C = \int_{X*Y*Z} \mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z)/(x,y,z)$$

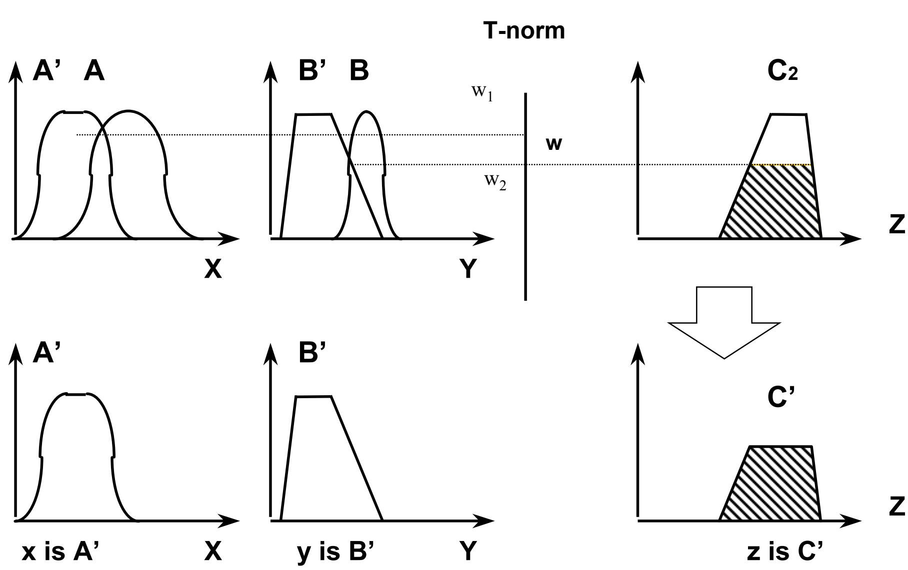
$$C' = \underbrace{(A'*B')}_{\text{premise 1}} \circ \underbrace{(A*B \rightarrow C)}_{\text{premise 2}}$$

$$\mu_{C'}(z) = \vee_{x,y} [\mu_{A'}(x) \wedge \mu_{B'}(y)] \wedge [\mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z)]$$

$$= \vee_{x,y} \{[\mu_{A'}(x) \wedge \mu_{B'}(y) \wedge \mu_A(x) \wedge \mu_B(y)]\} \wedge \mu_C(z)$$

$$= \underbrace{\left\{\vee_x [\mu_{A'}(x) \wedge \mu_A(x)]\right\}}_{w_1} \wedge \underbrace{\left\{\vee_y [\mu_{B'}(y) \wedge \mu_B(y)]\right\}}_{w2} \wedge \mu_C(z)$$

$$= (w_1 \wedge w_2) \wedge \mu_C(z)$$



Fuzzy Reasoning (3.4) (cont.)

– Multiple rules with multiple antecedents

Premise 1 (fact): x is A' and y is B'
Premise 2 (rule 1): if x is A_1 and y is B_1 then z is C_1
Premise 3 (rule 2): If x is A_2 and y is B_2 then z is C_2

Consequence (conclusion): z is C'

$R_1 = A_1 * B_1 \rightarrow C_1$
$R_2 = A_2 * B_2 \rightarrow C_2$
Since the max-min composition operator o is distributive over the union operator, it follows:
$C' = (A' * B') \circ (R_1 \cup R_2) = [(A' * B') \circ R_1] \cup [(A' * B') \circ R_2] = C'_1 \cup C'_2$
Where C'_1 & C'_2 are the inferred fuzzy set for rules 1 & 2 respectively