Chapter 3: Fuzzy Rules & Fuzzy Reasoning

- Extension Principle & Fuzzy Relations (3.2)
- Fuzzy if-then Rules (3.3)
- Fuzzy Reasoning (3.4)

Extension Principle & Fuzzy Relations (3.2)

Extension principle

A is a fuzzy set on X:

\[ A = \mu_A(x_1) / x_1 + \mu_A(x_2) / x_2 + \cdots + \mu_A(x_n) / x_n \]

The image of A under f(.) is a fuzzy set B:

\[ B = \mu_B(y_1) / y_1 + \mu_B(y_2) / y_2 + \cdots + \mu_B(y_n) / y_n \]

where \( y_i = f(x_i), \) i = 1 to n

If f(.) is a many-to-one mapping, then

\[ \mu_B(y) = \max_{x \in f^{-1}(y)} \mu_A(x) \]
Extension Principle & Fuzzy Relations (3.2) (cont.)

- **Example:**

  Application of the extension principle to fuzzy sets with discrete universes

  Let $A = 0.1 / -2 + 0.4 / -1 + 0.8 / 0 + 0.9 / 1 + 0.3 / 2$
  and $f(x) = x^2 - 3$

  Applying the extension principle, we obtain:
  
  $B = 0.1 / 1 + 0.4 / -2 + 0.8 / -3 + 0.9 / -2 + 0.3 / 1$
  
  $= 0.8 / -3 + (0.4V0.9) / -2 + (0.1V0.3) / 1$
  
  where “V” represents the “max” operator

 Same reasoning for continuous universes

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**Fuzzy relations**

- A fuzzy relation $R$ is a 2D MF:

  $$R = \{(x, y), \mu_R(x, y) | (x, y) \in X \times Y\}$$

  **Examples:**

  Let $X = Y = \mathbb{IR}^+$
  and $R(x, y) = "y \text{ is much greater than } x"$

  The MF of this fuzzy relation can be subjectively defined as:

  $$\mu_R(x, y) = \begin{cases} 
  \frac{y-x}{x+y+2}, & \text{if } y > x \\
  0, & \text{if } y \leq x
  \end{cases}$$
Extension Principle & Fuzzy Relations (3.2) (cont.)

• Then R can be written as a matrix:

\[
R = \begin{bmatrix}
0 & 0.111 & 0.200 & 0.273 & 0.333 \\
0 & 0 & 0.091 & 0.167 & 0.231 \\
0 & 0 & 0 & 0.077 & 0.143
\end{bmatrix}
\]

where \( R(i,j) = \mu[\xi,\eta] \)

- \( x \) is close to \( y \) (\( x \) and \( y \) are numbers)
- \( x \) depends on \( y \) (\( x \) and \( y \) are events)
- \( x \) and \( y \) look alike (\( x \) and \( y \) are persons or objects)
- If \( x \) is large, then \( y \) is small (\( x \) is an observed reading and \( Y \) is a corresponding action)

Extension Principle & Fuzzy Relations (3.2) (cont.)

- Max-Min Composition

• The max-min composition of two fuzzy relations \( R_1 \) (defined on \( X \) and \( Y \)) and \( R_2 \) (defined on \( Y \) and \( Z \)) is

\[
\mu_{R_1 \circ R_2}(x, z) = \bigvee_y \left[ \mu_{R_1}(x, y) \land \mu_{R_2}(y, z) \right]
\]

• Properties:
  - Associativity: \( R \circ (S \circ T) = (R \circ S) \circ T \)
  - Distributivity over union: \( R \circ (S \cup T) = (R \circ S) \cup (R \circ T) \)
  - Weak distributivity over intersection:
    \( R \circ (S \cap T) \subseteq (R \circ S) \cap (R \circ T) \)
  - Monotonicity: \( S \subseteq T \Rightarrow (R \circ S) \subseteq (R \circ T) \)
Extension Principle & Fuzzy Relations (3.2) (cont.)

- Max-min composition is not mathematically tractable, therefore other compositions such as max-product composition have been suggested

- Max-product composition

\[ \mu_{R_1 \circ R_2}(x, z) = \bigvee_y [\mu_{R_1}(x, y) \cdot \mu_{R_2}(y, z)] \]

Extension Principle & Fuzzy Relations (3.2) (cont.)

- Example of max-min & max-product composition

- Let \( R_1 = \text{“} x \text{ is relevant to } y \text{”} \)
  \( R_2 = \text{“} y \text{ is relevant to } z \text{”} \)

be two fuzzy relations defined on \( X \times Y \) and \( Y \times Z \) respectively
\( X = \{1,2,3\}, Y = \{\alpha, \beta, \chi, \delta\} \) and \( Z = \{a, b\} \).

Assume that:

\[
R_1 = \begin{bmatrix}
0.1 & 0.3 & 0.5 & 0.7 \\
0.4 & 0.2 & 0.8 & 0.9 \\
0.6 & 0.8 & 0.3 & 0.2
\end{bmatrix} \quad R_2 = \begin{bmatrix}
0.9 & 0.1 \\
0.2 & 0.3 \\
0.5 & 0.6 \\
0.7 & 0.2
\end{bmatrix}
\]
The derived fuzzy relation “x is relevant to z” based on $R_1$ & $R_2$

Let’s assume that we want to compute the degree of relevance between $2 \in X$ & $a \in Z$

Using max-min, we obtain:

$$\mu_{R_1 \circ R_2}(2, a) = \max\{0.4 \land 0.9, 0.2 \land 0.8, 0.5\} \land 0.7$$

$$= \max\{0.4, 0.2, 0.5, 0.7\}$$

$$= 0.7$$

Using max-product composition, we obtain:

$$\mu_{R_1 \circ R_2}(2, a) = \max\{0.4 \ast 0.9, 0.2 \ast 0.8, 0.5\} \ast 0.7$$

$$= \max\{0.36, 0.04, 0.40, 0.63\}$$

$$= 0.63$$

Linguistic Variables

- Conventional techniques for system analysis are intrinsically unsuited for dealing with systems based on human judgment, perception & emotion

- Principle of incompatibility
  - As the complexity of a system increases, our ability to make precise & yet significant statements about its behavior decreases until a fixed threshold
  - Beyond this threshold, precision & significance become almost mutually exclusive characteristics [Zadeh, 1973]
Fuzzy if-then rules (3.3) (cont.)

- The concept of linguistic variables introduced by Zadeh is an alternative approach to modeling human thinking.

- Information is expressed in terms of fuzzy sets instead of crisp numbers.

- **Definition:** A linguistic variable is a quintuple \((x, T(x), X, G, M)\) where:
  
  - \(x\) is the name of the variable
  - \(T(x)\) is the set of linguistic values (or terms)
  - \(X\) is the universe of discourse
  - \(G\) is a syntactic rule that generates the linguistic values
  - \(M\) is a semantic rule which provides meanings for the linguistic values

Fuzzy if-then rules (3.3) (cont.)

- **Example:**

  A numerical variable takes numerical values
  
  \[\text{Age} = 65\]

  A linguistic variables takes linguistic values
  
  \[\text{Age is old}\]

  A linguistic value is a fuzzy set

  All linguistic values form a term set

  \[T(\text{age}) = \{\text{young, not young, very young, ...} \]
  
  \[\text{middle aged, not middle aged, ...}\]
  
  \[\text{old, not old, very old, more or less old, ...}\]
  
  \[\text{not very yound and not very old, ...}\]
• Where each term $T(\text{age})$ is characterized by a fuzzy set of a universe of discourse $X = [0,100]$

Fuzzy if-then rules (3.3) (cont.)

– The syntactic rule refers to the way the terms in $T(\text{age})$ are generated

– The semantic rule defines the membership function of each linguistic value of the term set

– The term set consists of primary terms as (young, middle aged, old) modified by the negation (“not”) and/or the hedges (very, more or less, quite, extremely,...) and linked by connectives such as (and, or, either, neither,...)
Fuzzy if-then rules (3.3) (cont.)

Concentration & dilation of linguistic values

- Let $A$ be a linguistic value described by a fuzzy set with membership function $\mu_A(.)$
  
  $$A^k = \frac{\int \mu_A(x)^k}{x}$$

  is a modified version of the original linguistic value.

- $A^2 = \text{CON}(A)$ is called the concentration operation

- $\sqrt{A} = \text{DIL}(A)$ is called the dilation operation

- $\text{CON}(A)$ & $\text{DIL}(A)$ are useful in expression the hedges such as “very” & “more or less” in the linguistic term $A$

- Other definitions for linguistic hedges are also possible

Fuzzy if-then rules (3.3) (cont.)

Composite linguistic terms

Let’s define:

$$\text{NOT}(A) = \neg A = \frac{\int [1 - \mu_A(x)]}{x},$$

$$A \text{ and } B = A \cap B = \frac{\int [\mu_A(x) \land \mu_B(x)]}{x},$$

$$A \text{ or } B = A \cup B = \frac{\int [\mu_A(x) \lor \mu_B(x)]}{x},$$

where $A$, $B$ are two linguistic values whose semantics are respectively defined by $\mu_A(.)$ & $\mu_B(.)$

Composite linguistic terms such as: “not very young”, “not very old” & “young but not too young” can be easily characterized
Fuzzy if-then rules (3.3) (cont.)

– Example: Construction of MFs for composite linguistic terms

Let’s

\[ \mu_{\text{young}}(x) = \text{bell}(x, 20, 2, 0) = \frac{1}{1 + \left(\frac{x}{20}\right)^4} \]

\[ \mu_{\text{old}}(x) = \text{bell}(x, 30, 3, 100) = \frac{1}{1 + \left(\frac{x - 100}{30}\right)^6} \]

Where \( x \) is the age of a person in the universe of discourse \([0, 100]\)

• More or less = DIIL(old) = \( \wedge \text{old} = \int_{x} \frac{1}{1 + \left(\frac{x - 100}{30}\right)^6} / x \)

Fuzzy if-then rules (3.3) (cont.)

• Not young and not old = \( \neg \text{young} \wedge \neg \text{old} = \int_{x} \left[ \frac{1}{1 + \left(\frac{x}{20}\right)^4} \right] \wedge \left[ 1 - \frac{1}{1 + \left(\frac{x - 100}{30}\right)^6} \right] / x \)

• Young but not too young = young \wedge \neg \text{young}^2 \text{ (too = very) } = \int_{x} \left[ \frac{1}{1 + \left(\frac{x}{20}\right)^4} \right] ^2 \wedge \left[ 1 - \left(\frac{1}{1 + \left(\frac{x}{20}\right)^4}\right)^2 \right] / x \)

• Extremely old \equiv \text{very very very old} = \text{CON} (\text{CON(\text{CON(old)})}) = \int_{x} \left[ \frac{1}{1 + \left(\frac{x - 100}{30}\right)^6} \right]^8 / x
- Contrast intensification

the operation of contrast intensification on a linguistic value A is defined by

$$\text{INT}(A) = \begin{cases} 2A^2 & \text{if } 0 \leq \mu_A(x) \leq 0.5 \\ -2(-A)^2 & \text{if } 0.5 \leq \mu_A(x) \leq 1 \end{cases}$$

- INT increases the values of $\mu_a(x)$ which are greater than 0.5 & decreases those which are less or equal that 0.5

- Contrast intensification has effect of reducing the fuzziness of the linguistic value A
A term set $T = t_1, \ldots, t_n$ of a linguistic variable $x$ on the universe $X$ is orthogonal if:

$$\sum_{i=1}^{n} \mu_{t_i}(x) = 1, \quad \forall x \in X$$

Where the $t_i$'s are convex & normal fuzzy sets defined on $X$. 

Fuzzy if-then rules (3.3) (cont.)

– Orthogonality
Fuzzy if-then rules (3.3) (cont.)

General format:

– If \( x \) is \( A \) then \( y \) is \( B \) (where \( A \) & \( B \) are linguistic values defined by fuzzy sets on universes of discourse \( X \) & \( Y \)).

  • “\( x \) is \( A \)” is called the antecedent or premise
  • “\( y \) is \( B \)” is called the consequence or conclusion

– Examples:

  • If pressure is high, then volume is small.
  • If the road is slippery, then driving is dangerous.
  • If a tomato is red, then it is ripe.
  • If the speed is high, then apply the brake a little.

Fuzzy if-then rules (3.3) (cont.)

– Meaning of fuzzy if-then-rules \((A \Rightarrow B)\)

  • It is a relation between two variables \( x \) & \( y \); therefore it is a binary fuzzy relation \( R \) defined on \( X \times Y \)

  • There are two ways to interpret \( A \Rightarrow B \):

    – A coupled with B
    – A entails B

    if \( A \) is coupled with \( B \) then:

    \[
    R = A \Rightarrow B = A \ast B = \int_{X \times Y} \mu_A(x) \ast \mu_B(y)/(x, y)
    \]

    where \( \ast \) is a \( T \)-norm operator.
Fuzzy if-then rules (3.3) (cont.)

If $A$ entails $B$ then:

\[ R = A \Rightarrow B = \neg A \cup B \] (material implication)

\[ R = A \Rightarrow B = \neg A \cup (A \cap B) \] (propositional calculus)

\[ R = A \Rightarrow B = (\neg A \cap \neg B) \cup B \] (extended propositional calculus)

\[
\mu_R(x, y) = \sup \left\{ c; \mu_A(x)^*c \leq \mu_B(y), 0 \leq c \leq 1 \right\}
\]

Fuzzy if-then rules (3.3) (cont.)

Two ways to interpret “If $x$ is $A$ then $y$ is $B$”:

- $A$ coupled with $B$

- $A$ entails $B$
Fuzzy if-then rules (3.3) (cont.)

- Note that $R$ can be viewed as a fuzzy set with a two-dimensional MF

$$\mu_R(x, y) = f(\mu_A(x), \mu_B(y)) = f(a, b)$$

With $a = \mu_A(x)$, $b = \mu_B(y)$ and $f$ called the fuzzy implication function provides the membership value of $(x, y)$

Fuzzy if-then rules (3.3) (cont.)

- Case of "A coupled with B"

$$R_e = A \ast_B B = \int_{X \times Y} \mu_A(x) \mu_B(y) dx dy$$

(minimum operator proposed by Mamdani, 1975)

$$R_p = A \ast_B B = \int_{X \times Y} \mu_A(x) \mu_B(y) dx dy$$

(product proposed by Larsen, 1980)

$$R_{bp} = A \ast_B B = \int_{X \times Y} \mu_A(x) \bigotimes \mu_B(y) dx dy$$

$$= \int_{X \times Y} 0 \lor (\mu_A(x) + \mu_B(y) - 1) dx dy$$

(bounded product operator)
Fuzzy if-then rules (3.3) (cont.)

- Case of “A coupled with B” (cont.)

\[ R_{ab} = A \ast B = \int_{x,y} \mu_a(x) \cdot \mu_b(y) / (x, y) \]

where \( f(a, b) = a \ast b = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{if otherwise} \end{cases} \)

Example for \( \mu_a(x) = \text{bell}(x; 4, 3, 10) \) and \( \mu_b(y) = \text{bell}(y; 4, 3, 10) \)

(Drastic operator)

A coupled with B

CH. 3: Fuzzy rules & fuzzy reasoning
Fuzzy if-then rules (3.3) (cont.)

– Case of “A entails B”

\[ R_a = \neg A \cup B = \int_{X \times Y} 1 \wedge (1 - \mu_A(x) + \mu_B(y))/(x,y) \]

where : \( f_a(a,b) = 1 \wedge (1 - a + b) \)

(Zadeh's arithmetic rule by using bounded sum operator for union)

\[ R_{nm} = \neg A \cup (A \cap B) = \int_{X \times Y} (1 - \mu_A(x)) \vee (\mu_A(x) \wedge \mu_B(y))/(x,y) \]

where : \( f_m(a,b) = (1 - a) \vee (a \wedge b) \)

(Zadeh's max-min rule)

Fuzzy if-then rules (3.3) (cont.)

– Case of “A entails B” (cont.)

\[ R_s = \neg A \cup B = \int_{X \times Y} (1 - \mu_A(x)) \vee \mu_B(y)/(x,y) \]

where : \( f_s(a,b) = (1 - a) \vee b \)

(Boolean fuzzy implication with max for union)

\[ R_A = \int_{X \times Y} (\mu_A(x) \lesssim \mu_B(y))/(x,y) \]

where : \( a \lesssim b = \begin{cases} 1 & \text{if } a \leq b \\ b/a & \text{otherwise} \end{cases} \)

(Goguen's fuzzy implication with algebraic product for T-norm)
Fuzzy Reasoning (3.4)

**Definition**

- Known also as approximate reasoning

- It is an inference procedure that derives conclusions from a set of fuzzy if-then-rules & known facts

A entails B
Fuzzy Reasoning (3.4) (cont.)

- Compositional rule of inference
  - Idea of composition (cylindrical extension & projection)
    - Computation of b given a & f is the goal of the composition
      - Image of a point is a point
      - Image of an interval is an interval

Derivation of $y = b$ from $x = a$ and $y = f(x)$:

- **Points**: $a$ and $b$ are points.
  - $y = f(x)$ is a curve.

- **Intervals**: $a$ and $b$ are intervals.
  - $y = f(x)$ is an interval-valued function.
Fuzzy Reasoning (3.4) (cont.)

- The extension principle is a special case of the compositional rule of inference

  - $F$ is a fuzzy relation on $X^*Y$, $A$ is a fuzzy set of $X$ & the goal is to determine the resulting fuzzy set $B$

  - Construct a cylindrical extension $c(A)$ with base $A$

  - Determine $c(A) \wedge F$ (using minimum operator)

  - Project $c(A) \wedge F$ onto the y-axis which provides $B$

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**a** is a fuzzy set and $y = f(x)$ is a fuzzy relation:
Given A, A \implies B, infer B

A = “today is sunny”
A \implies B: day = sunny then sky = blue
infer: “sky is blue”

- illustration

Premise 1 (fact): \ x is A
Premise 2 (rule): \ if \ x \ is \ A \ then \ y \ is \ B

Consequence: \ y \ is \ B

Approximation

A’ = “today is more or less sunny”
B’ = “sky is more or less blue”

- illustration

Premise 1 (fact): \ x \ is \ A’
Premise 2 (rule): \ if \ x \ is \ A \ then \ y \ is \ B

Consequence: \ y \ is \ B’

(approximate reasoning or fuzzy reasoning!)
Fuzzy Reasoning (3.4) (cont.)

Definition of fuzzy reasoning

Let A, A’ and B be fuzzy sets of X, X, and Y, respectively. Assume that the fuzzy implication 
$A \Rightarrow B$ is expressed as a fuzzy relation R on $X \times Y$. Then 
the fuzzy set B induced by “x is A’ ” and the fuzzy rule 
“If x is A then y is B’ is defined by:

$$
\mu_{B'}(y) = \max \min_{x} [\mu_{A'}(x), \mu_{R}(x, y)] 
$$

Single rule with single antecedent

Rule : if x is A then y is B
Fact: x is A’
Conclusion: y is B’ ($\mu_{B'}(y) = [\vee_{x} (\mu_{A}(x) \land \mu_{A}(x)) \land \mu_{B}(y)]$)
– Single rule with multiple antecedents
Premise 1 (fact): x is A' and y is B'
Premise 2 (rule): if x is A and y is B then z is C

Conclusion: z is C'

Premise 2: A*B → C

\[
R_{\text{mandani}}(A, B, C) = (A * B) * C = \bigwedge_{X^*Y^*Z} \mu_A(x) \land \mu_B(y) \land \mu_C(z)/(x, y, z)
\]

\[
C' = (A'^*B') \circ (A * B \rightarrow C)
\]

\[
\mu_{C'}(z) = \bigvee_{x,y} \left[ \mu_{A'}(x) \land \mu_{B'}(y) \right] \land \left[ \mu_A(x) \land \mu_B(y) \land \mu_C(z) \right]
\]

\[
= \bigvee_{x,y} \left[ \mu_{A'}(x) \land \mu_{B'}(y) \right] \land \mu_A(x) \land \mu_B(y) \land \mu_C(z)
\]

\[
= \left\{ \bigvee_{x} \left[ \mu_{A'}(x) \land \mu_A(x) \right] \right\} \land \left\{ \bigvee_{y} \left[ \mu_{B'}(y) \land \mu_B(y) \right] \right\} \land \mu_C(z)
\]

\[
= (w_1 \land w_2) \land \mu_C(z)
\]
Fuzzy Reasoning (3.4) (cont.)

- Multiple rules with multiple antecedents

Premise 1 (fact): \( x \) is \( A' \) and \( y \) is \( B' \)
Premise 2 (rule 1): if \( x \) is \( A_1 \) and \( y \) is \( B_1 \) then \( z \) is \( C_1 \)
Premise 3 (rule 2): If \( x \) is \( A_2 \) and \( y \) is \( B_2 \) then \( z \) is \( C_2 \)

Consequence (conclusion): \( z \) is \( C' \)

\[
R_1 = A_1 \times B_1 \rightarrow C_1 \\
R_2 = A_2 \times B_2 \rightarrow C_2
\]

Since the max-min composition operator \( o \) is distributive over the union operator, it follows:

\[
C' = (A' \times B') \circ (R_1 \cup R_2) = [(A' \times B') \circ R_1] \cup [(A' \times B') \circ R_2] = C'_1 \cup C'_2
\]

Where \( C'_1 \) & \( C'_2 \) are the inferred fuzzy set for rules 1 & 2 respectively