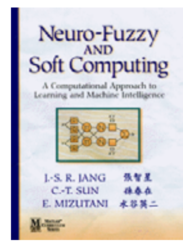
Chapter 2: FUZZY SETS

- Introduction (2.1)
- Basic Definitions &Terminology (2.2)
- Set-theoretic Operations (2.3)
- Membership Function (MF) Formulation & Parameterization (2.4)
- More on Fuzzy Union, Intersection & Complement (2.5)

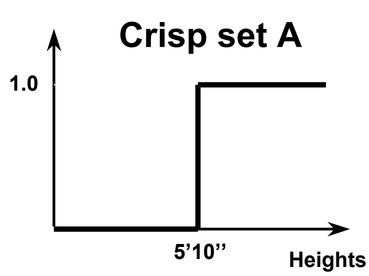


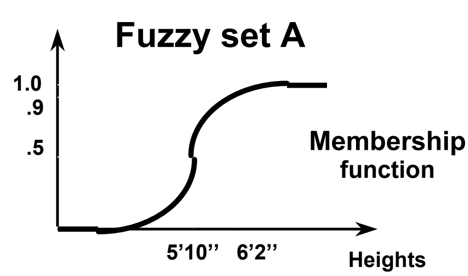
Jyh-Shing Roger Jang et al., *Neuro-Fuzzy and Soft Computing: A Computational Approach to Learning and Machine Intelligence*, First Edition, Prentice Hall, 1997

Introduction (2.1)

- Sets with fuzzy boundaries

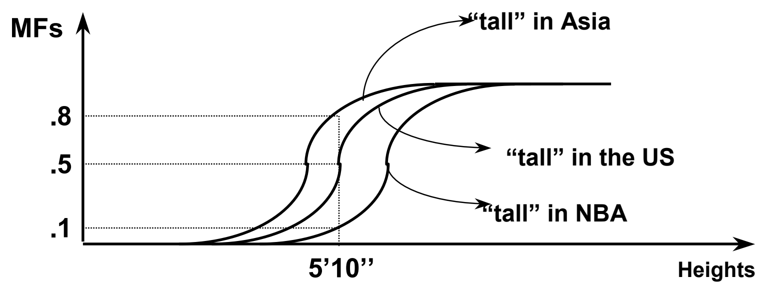
A = Set of tall people





Introduction (2.1) (cont.)

- Membership Functions (MFs)
  - Characteristics of MFs:
    - Subjective measures
    - Not probability functions



Basic definitions & Terminology (2.2)

- Formal definition:

A fuzzy set *A* in *X* is expressed as a set of ordered pairs:

$$A = \{(x, \mu_A(x)) | x \in X\}$$

Fuzzy set — **Membership function (MF)** — **Universe or universe of discourse**

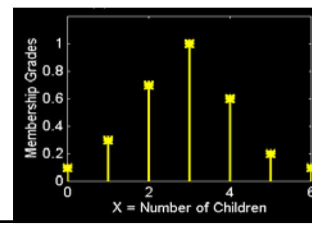
A fuzzy set is totally characterized by a membership function (MF).

Basic definitions & Terminology (2.2) (cont.)

Fuzzy Sets with Discrete Universes

- Fuzzy set C = “desirable city to live in”
  X = {SF, Boston, LA} (discrete and non-ordered)
  C = {(SF, 0.9), (Boston, 0.8), (LA, 0.6)}
  (subjective membership values!)

- Fuzzy set A = “sensible number of children”
  X = {0, 1, 2, 3, 4, 5, 6} (discrete universe)
  A = {(0, .1), (1, .3), (2, .7), (3, 1), (4, .6), (5, .2), (6, .1)}
  (subjective membership values!)

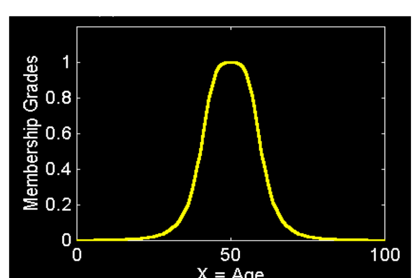


Basic definitions & Terminology (2.2) (cont.)

Fuzzy Sets with Cont. Universes

- Fuzzy set B = “about 50 years old”
  X = Set of positive real numbers (continuous)
  B = {(x, μ_B(x)) | x in X}

$$\mu_B(x) = \frac{1}{1 + \left(\frac{x - 50}{10}\right)^2}$$



Basic definitions & Terminology (2.2) (cont.)

◆Alternative Notation

◆A fuzzy set A can be alternatively denoted as follows:

X is discrete ➡ $$A = \sum_{x_i \in X} \mu_A(x_i) / x_i$$

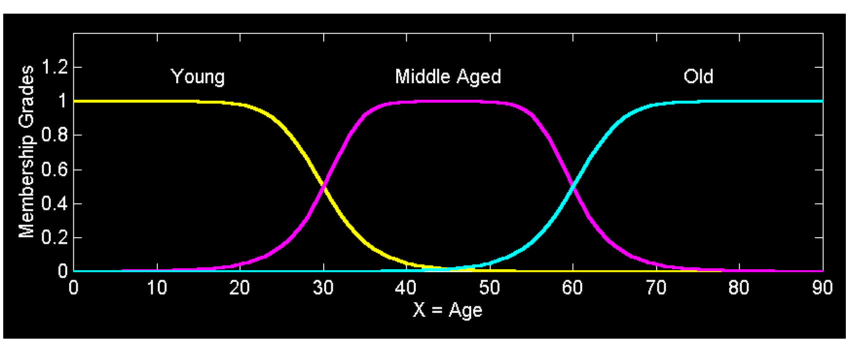
X is continuous ➡ $$A = \int_X \mu_A(x) / x$$

Note that Σ and integral signs stand for the union of membership grades; "/" stands for a marker and does not imply division.

Basic definitions & Terminology (2.2) (cont.)

◆Fuzzy Partition

◆ Fuzzy partitions formed by the linguistic values
◆ "young", "middle aged", and "old":

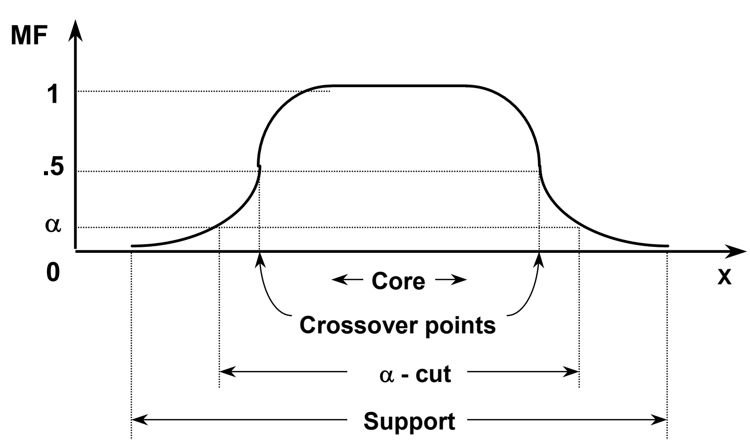


Basic definitions & Terminology (2.2) (cont.)

- Support(A) = $\{x \in X \mid \mu_A(x) > 0\}$
- Core(A) = $\{x \in X \mid \mu_A(x) = 1\}$
- Normality: core(A) $\neq \emptyset \Rightarrow$ A is a normal fuzzy set
- Crossover(A) = $\{x \in X \mid \mu_A(x) = 0.5\}$
- α - cut: $A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$
- Strong α - cut: $A'_\alpha = \{x \in X \mid \mu_A(x) > \alpha\}$

Basic definitions & Terminology (2.2) (cont.)

MF Terminology

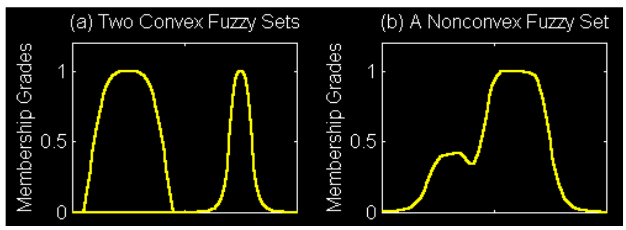


Basic definitions & Terminology (2.2) (cont.)

◆ Convexity of Fuzzy Sets

◆A fuzzy set *A* is convex if for any λ in [0, 1],

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$



Basic definitions & Terminology (2.2) (cont.)

◆Fuzzy numbers: a fuzzy number A is a fuzzy set in IR that satisfies normality & convexity

◆Bandwidths: for a normal & convex set, the bandwidth is the distance between two unique crossover points

$$\text{Width}(A) = |x_2 - x_1|$$

$$\text{With } \mu_A(x1) = \mu_A(x2) = 0.5$$

◆Symmetry: a fuzzy set A is symmetric if its MF is symmetric around a certain point x = c, namely

$$\mu_A(x + c) = \mu_A(c - x) \quad \forall x \in X$$

Basic definitions & Terminology (2.2) (cont.)

- Open left, open right, closed:

$$\text{open left fuzzy set } A \Leftrightarrow \lim_{x\to-\infty} \mu_A(x) = 1 \text{ and } \lim_{x\to+\infty} \mu_A(x) = 0$$

$$\text{open right fuzzy set } A \Leftrightarrow \lim_{x\to-\infty} \mu_A(x) = 0 \text{ and } \lim_{x\to+\infty} \mu_A(x) = 1$$

$$\text{closed fuzzy set } A \Leftrightarrow \lim_{x\to-\infty} \mu_A(x) = \lim_{x\to+\infty} \mu_A(x) = 0$$

Set-Theoretic Operations (2.3)

- Subset:

$$A \subseteq B \Leftrightarrow \mu_A \le \mu_B$$

- Complement:

$$\overline{A} = X - A \Leftrightarrow \mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

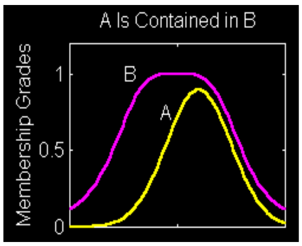
- Union:

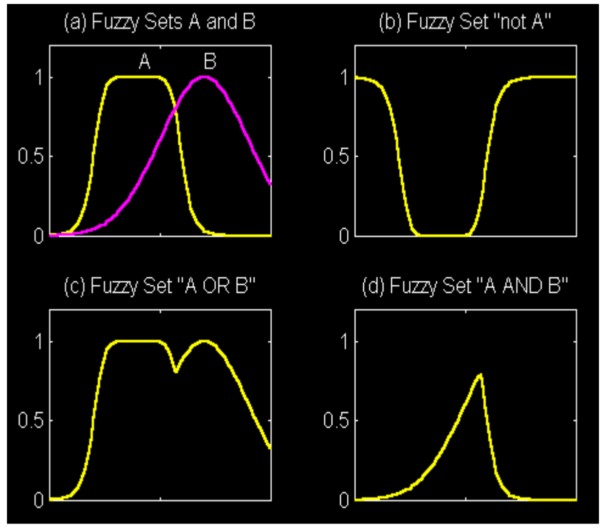
$$C = A \cup B \Leftrightarrow \mu_c(x) = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \vee \mu_B(x)$$

- Intersection:

$$C = A \cap B \Leftrightarrow \mu_c(x) = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \wedge \mu_B(x)$$

Set-Theoretic Operations (2.3) (cont.)



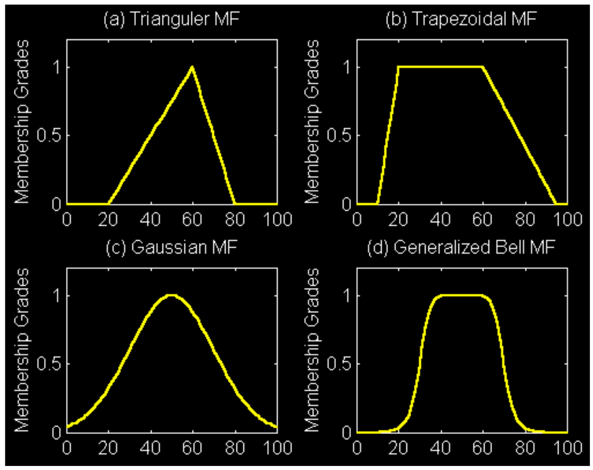


MF Formulation & Parameterization (2.4)

MFs of One Dimension

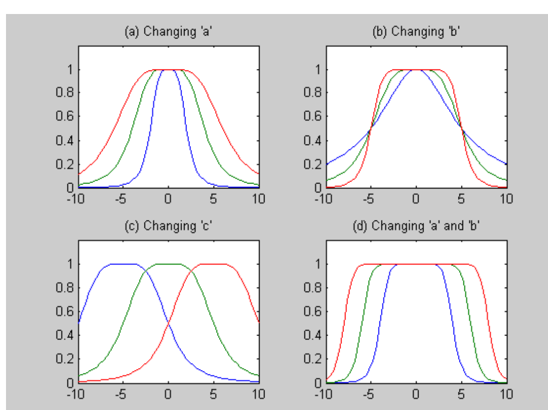
- Triangular MF: $trimf(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$
- Trapezoidal MF: $trapmf(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$
- Gaussian MF: $gaussmf(x;c,\sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$
- Generalized bell MF: $gbellmf(x;a,b,c) = \frac{1}{1+\left|\frac{x-c}{b}\right|^{2b}}$

MF Formulation & Parameterization (2.4) (cont.)

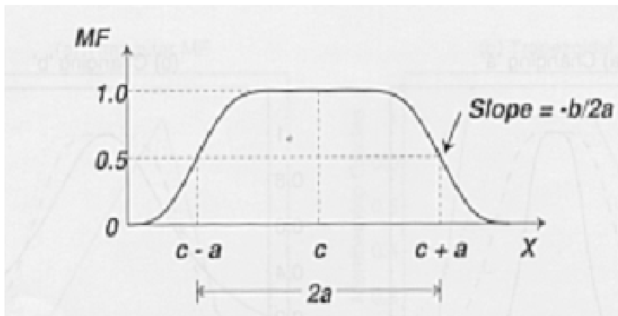


MF Formulation & Parameterization (2.4) (cont.)

- Change of parameters in the generalized bell MF



MF Formulation & Parameterization (2.4) (cont.)



Physical meaning of parameters in a generalized bell MF

MF Formulation & Parameterization (2.4) (cont.)

- Gaussian MFs and bell MFs achieve smoothness, they are unable to specify asymmetric Mfs which are important in many applications
- Asymmetric & close MFs can be synthesized using either the absolute difference or the product of two sigmoidal functions

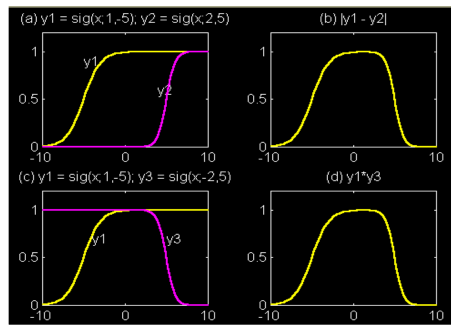
MF Formulation & Parameterization (2.4) (cont.)

- Sigmoidal MF: $$\text{sigmf}(x; a, c) = \frac{1}{1 + e^{-a(x-c)}}$$

Extensions:

Abs. difference of two sig. MF →

Product of two sig. MF →



MF Formulation & Parameterization (2.4) (cont.)

- A sigmoidal MF is inherently open right or left & thus, it is appropriate for representing concepts such as “very large” or “very negative”
- Sigmoidal MF mostly used as activation function of artificial neural networks (NN)
- A NN should synthesize a close MF in order to simulate the behavior of a fuzzy inference system

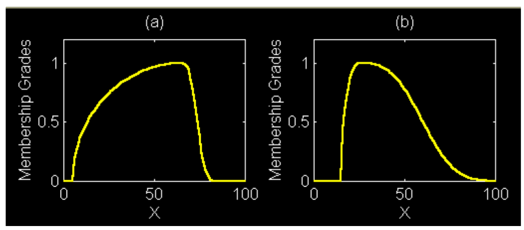
MF Formulation & Parameterization (2.4) (cont.)

- Left –Right (LR) MF:

$$LR(x; c, \alpha, \beta) = \begin{cases} F_L\left(\frac{c-x}{\alpha}\right), x < c \\ F_R\left(\frac{x-c}{\beta}\right), x \geq c \end{cases}$$

Example: $F_L(x) = \sqrt{\max(0, 1 - x^2)}$ $F_R(x) = \exp(-|x|^3)$

c=65
a=60
b=10



c=25
a=10
b=40

MF Formulation & Parameterization (2.4) (cont.)

- The list of MFs introduced in this section is by no means exhaustive
- Other specialized MFs can be created for specific applications if necessary
- Any type of continuous probability distribution functions can be used as an MF

MF Formulation & Parameterization (2.4) (cont.)

- MFs of two dimensions
  - In this case, there are two inputs assigned to an MF: this MF is a twp dimensional MF. A one input MF is called ordinary MF
  - Extension of a one-dimensional MF to a two-dimensional MF via cylindrical extensions
    - If A is a fuzzy set in X, then its cylindrical extension in X*Y is a fuzzy set C(A) defined by:

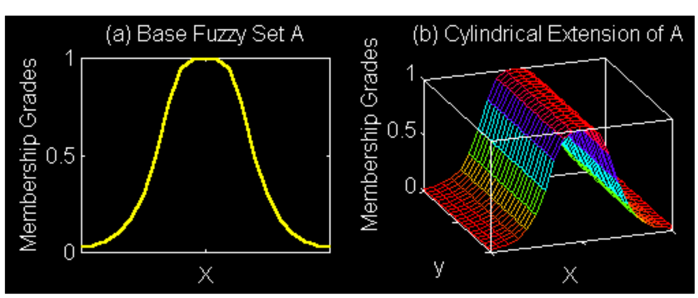
$$C(A) = \int_{X*Y} \mu_A(x) \,|\, (x, y)$$

    - C(A) can be viewed as a two-dimensional fuzzy set

MF Formulation & Parameterization (2.4) (cont.)

Cylindrical extension

Base set A **Cylindrical Ext. of A**



MF Formulation & Parameterization (2.4) (cont.)

- Projection of fuzzy sets (decrease dimension)
  - Let R be a two-dimensional fuzzy set on X*Y. Then the projections of R onto X and Y are defined as:

$$R_X = \int_X \left[\max_y \mu_R(x,y) \right] | x$$

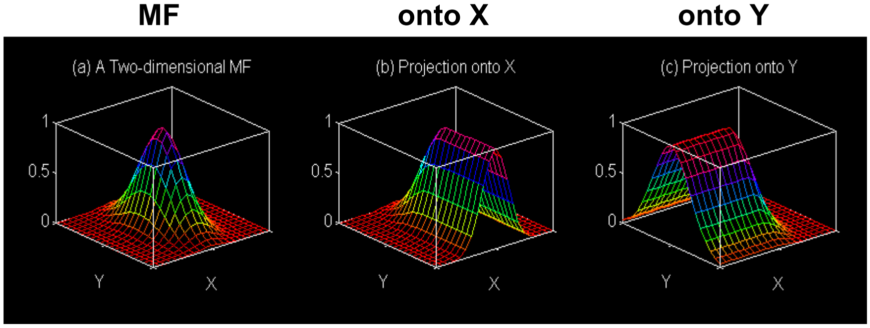
and

$$R_Y = \int_Y \left[\max_x \mu_R(x,y) \right] | y$$

respectively.

MF Formulation & Parameterization (2.4) (cont.)

Two-dimensional MF	Projection onto X	Projection onto Y



MF Formulation & Parameterization (2.4) (cont.)

- Composite & non-composite MFs
  - Suppose that the fuzzy A = "(x,y) is near (3,4)" is defined by:

$$\begin{aligned}\mu_A(x,y) &= \exp\left[-\left(\frac{x-3}{2}\right)^2 - (y-4)^2\right] \\ &= \exp\left[-\left(\frac{x-3}{2}\right)^2\right]\exp\left[-\left(\frac{y-4}{1}\right)^2\right] \\ &= G(x;3,2) * G(y;4,1)\end{aligned}$$

  - This two-dimensional MF is composite
  - The fuzzy set A is composed of two statements:

"x is near 3" & "y is near 4"

MF Formulation & Parameterization (2.4) (cont.)

- These two statements are respectively defined as:

$\mu_{\text{near } 3}(x) = G(x;3,2)$ & $\mu_{\text{near } 4}(x) = G(y;4,1)$

- If a fuzzy set is defined by:

$$\mu_A(x,y) = \frac{1}{1 + |x-3||y-4|^{2.5}}$$

it is non-composite.

- A composite two-dimensional MF is usually the result of two statements joined by the AND or OR connectives.

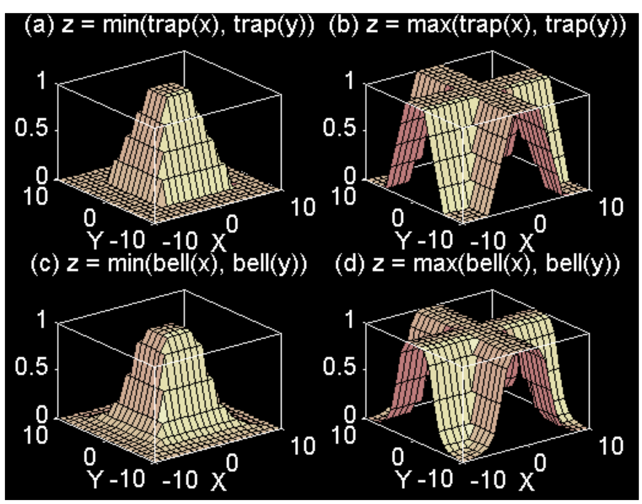
MF Formulation & Parameterization (2.4) (cont.)

- Composite two-dimensional MFs based on min & max operations

- Let trap(x) = trapezoid (x;-6,-2,2,6)
  trap(y) = trapezoid (y;-6,-2,2,6)
  be two trapezoidal MFs on X and Y respectively

- By applying the min and max operators, we obtain two-dimensional MFs on X*Y.

MF Formulation & Parameterization (2.4) (cont.)



Two dimensional MFs defined by the min and max operators

More on Fuzzy Union, Intersection & Complement (2.5)

- Fuzzy complement
  - Another way to define reasonable & consistent operations on fuzzy sets
    - General requirements:
      - Boundary: N(0)=1 and N(1) = 0
      - Monotonicity: N(a) > N(b) if a < b
      - Involution: N(N(a) = a

More on Fuzzy Union, Intersection & Complement (2.5) (cont.)

- Two types of fuzzy complements:
  - Sugeno's complement:

$$N_s(a) = \frac{1-a}{1+sa} \quad (s > -1)$$

(Family of fuzzy complement operators)

  - Yager's complement:

$$N_w(a) = (1-a^w)^{1/w} \quad (w > 0)$$

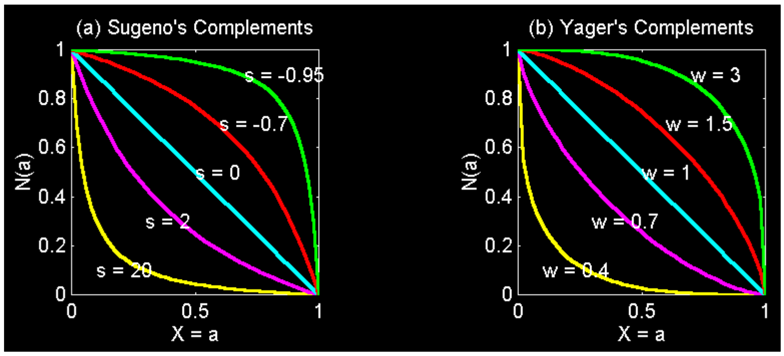
More on Fuzzy Union, Intersection & Complement (2.5) (cont.)

Sugeno's complement:

$$N_s(a) = \frac{1-a}{1+sa}$$

Yager's complement:

$$N_w(a) = (1-a^w)^{1/w}$$



More on Fuzzy Union, Intersection & Complement (2.5) (cont.)

- Fuzzy Intersection and Union:
  - The intersection of two fuzzy sets A and B is specified in general by a function

    T: [0,1] * [0,1] → [0,1] with

    $$\mu_{A \cap B}(x) = T(\mu_A(x), \mu_B(x)) = \mu_A(x) \tilde{*} \mu_B(x)$$

    where $\tilde{*}$ is a binary operator for the function T.

    This class of fuzzy intersection operators are called T-norm (triangular) operators.

More on Fuzzy Union, Intersection & Complement (2.5) (cont.)

- T-norm operators satisfy:
  - Boundary: $T(0, 0) = 0$, $T(a, 1) = T(1, a) = a$
    Correct generalization to crisp sets
  - Monotonicity: $T(a, b) < T(c, d)$ if $a < c$ and $b < d$
    A decrease of membership in A & B cannot increase a membership in $A \cap B$
  - Commutativity: $T(a, b) = T(b, a)$
    T is indifferent to the order of fuzzy sets to be combined
  - Associativity: $T(a, T(b, c)) = T(T(a, b), c)$
    Intersection is independent of the order of pairwise groupings

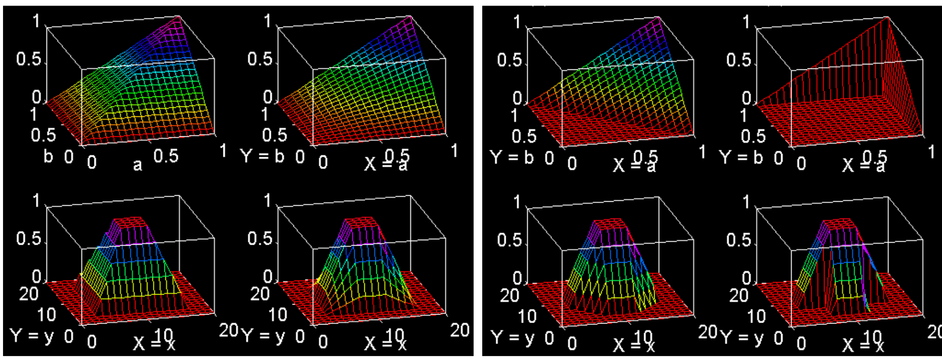
More on Fuzzy Union, Intersection & Complement (2.5) (cont.)

- T-norm (cont.)
  - Four examples (page 37):
    - Minimum: $T_m(a, b) = \min(a,b) = a \wedge b$
    - Algebraic product: $T_a(a, b) = ab$
    - Bounded product: $T_b(a, b) = 0 \vee (a + b - 1)$
    - Drastic product: $T_d(a, b) = \begin{cases} a, & \text{if } b = 1 \\ b, & \text{if } a = 1 \\ 0, & \text{if } a,b < 1 \end{cases}$

More on Fuzzy Union, Intersection & Complement (2.5) (cont.)

T-norm Operator

Minimum: $T_m(a, b)$	**Algebraic product:** $T_a(a, b)$	**Bounded product:** $T_b(a, b)$	**Drastic product:** $T_d(a, b)$



More on Fuzzy Union, Intersection & Complement (2.5) (cont.)

- T-conorm or S-norm

The fuzzy union operator is defined by a function

$$S: [0,1] * [0,1] \rightarrow [0,1]$$

wich aggregates two membership function as:

$$\mu_{A \cup B} = S(\mu_A(x), \mu_B(x)) = \mu_A(x) \tilde{+} \mu_B(x)$$

where s is called an s-norm satisfying:

- Boundary: S(1, 1) = 1, S(a, 0) = S(0, a) = a
- Monotonicity: S(a, b) < S(c, d) if a < c and b < d
- Commutativity: S(a, b) = S(b, a)
- Associativity: S(a, S(b, c)) = S(S(a, b), c)

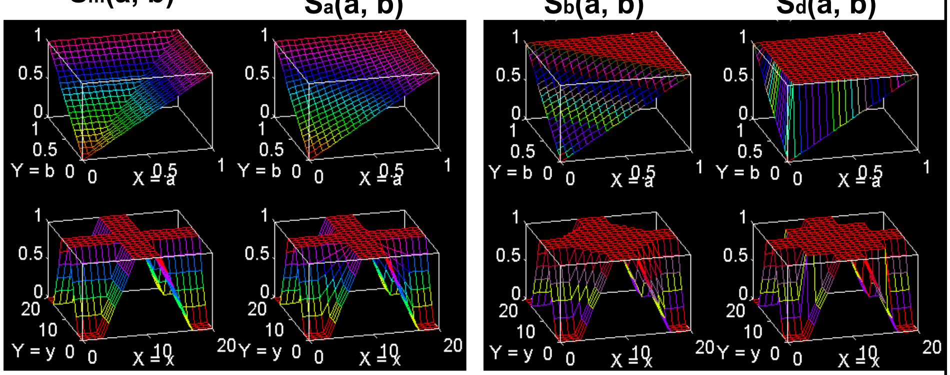
More on Fuzzy Union, Intersection & Complement (2.5) (cont.)

- T-conorm or S-norm (cont.)
  - Four examples (page 38):
    - Maximum: $S_m(a, b) = \max(a,b) = a \vee b$
    - Algebraic sum: $S_a(a, b) = a + b - ab$
    - Bounded sum: $S_b(a, b) = 1 \wedge (a + b)$
    - Drastic sum: $S_d(a, b) = \begin{cases} \mathbf{a}, & \mathbf{if\ b = 0} \\ \mathbf{b}, & \mathbf{if\ a = 0} \\ \mathbf{1}, & \mathbf{if\ a,b > 0} \end{cases}$

More on Fuzzy Union, Intersection & Complement (2.5) (cont.)

T-conorm or S-norm

Maximum: S_m(a, b)	**Algebraic sum: S_a(a, b)**	**Bounded sum: S_b(a, b)**	**Drastic sum: S_d(a, b)**



More on Fuzzy Union, Intersection & Complement (2.5) (cont.)

- Generalized DeMorgan's Law
  - T-norms and T-conorms are duals which support the generalization of DeMorgan's law:
    - T(a, b) = N(S(N(a), N(b)))
    - S(a, b) = N(T(N(a), N(b)))

T_m(a, b) ⟷ **S_m(a, b)**
T_a(a, b) ⟷ **S_a(a, b)**
T_b(a, b) ⟷ **S_b(a, b)**
T_d(a, b) ⟷ **S_d(a, b)**

More on Fuzzy Union, Intersection & Complement (2.5) (cont.)

- Parameterized T-norm and T-conorm
  - Parameterized T-norms and dual T-conorms have been proposed by several researchers:
    - Yager
    - Schweizer and Sklar
    - Dubois and Prade
    - Hamacher
    - Frank
    - Sugeno
    - Dombi