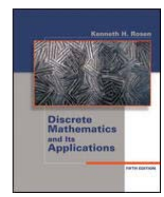
Chapter 7 (Part 2): Relations

- Representing Relations (7.3)
- Equivalence Relations (7.5)



Representing Relations (7.3)

- First way is to list the ordered pairs
- Second way is through matrices
- Third way is through direct graphs

Representing Relations (7.3)

- Representing relations through matrices
- $$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{otherwise} \end{cases}$$
  - Example: Suppose that the relation R on a set is represented by the matrix: $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$

    Is R reflexive, symmetric, and/or antisymmetric?

    Solution: Since all the diagonal elements of this matrix are equal to 1, R is reflexive. Moreover, since M_R is symmetric $\Rightarrow$ R is symmetric. R is not antisymmetric.

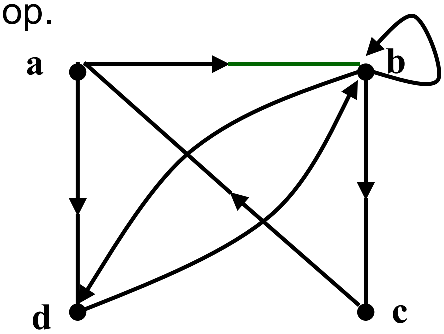
Representing Relations (7.3)

- Representing relations using diagraphs
  - Definition 1:

    A directed graph, or diagraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.

Representing Relations (7.3)

– Example: The directed graph with vertices a, b, c and d , and edges (a,b), (a,d), (b,b), (b,d), (c,a) and (d,b). The edge (b,b) is called a loop.



Equivalence Relations (7.5)

- Students registration time with respect to the first letter of their names
- R contains (x,y) ⇔ x and y are students with last names beginning with letters in the same block
- 3 blocks are considered: A-F, G-O, P-Z
- R is reflexive, symmetric & transitive
- The set of student is therefore divided in 3 classes depending on the first letter of their names

Equivalence Relations (7.5)

- Definition 1

> A relation on a set A is called an equivalence relation if it is reflexive, symmetric and transitive.

- Examples
  :
  - Suppose that R is the relation on the set of strings of English letters such that *aRb* if and only if l(a) = l(b), where l(x) is the length of the string x. Is R an equivalence relation?
    Solution: R is reflexive, symmetric and transitive $\Rightarrow$ R is an equivalence relation
  - A divides b; is it an equivalence relation?

Equivalence Relations (7.5)

- Equivalence classes
  - Definition 2:

> Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the equivalence class of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we will delete the subscript R and write [a] for this equivalence class.

Equivalence Relations (7.5)

– Example: What are the equivalences classes of 0 and 1 for congruence modulo 4?

Solution:
The equivalence class of 0 contains all the integers a such that $a \equiv 0 \pmod 4$. Hence, the equivalence class of 0 for this relation is

$$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$$

The equivalence class of 1 contains all the integers a such that $a \equiv 1 \pmod 4$. The integers in this class are those that have a remainder of 1 when divided by 4. Hence, the equivalence class of 1 for this relation is

$$[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$$

Equivalence Relations (7.5)

- Equivalence classes & partitions

  – Theorem 1:

> Let R be an equivalence relation on a set A. These statements are equivalent:
> i. a R b
> ii. $[a] = [b]$
> iii. $[a] \cap [b] \neq \emptyset$

Equivalence Relations (7.5)

– Theorem 2:

> Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i , $i \in I$, as its equivalence classes.

Equivalence Relations (7.5)

– Example: List the ordered pairs in the equivalence relation R produced by the partition A_1 = [1,2,3}, A_2 = {4,5} and A_3 = {6} of S = {1,2,3,4,5,6}

Solution: The subsets in the partition are the equivalences classes of R. The pair $(a,b) \in R$ if and only if a and b are in the same subset of the partition.

The pairs (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2) and $(3,3) \in R \Leftarrow A_1$ = [1,2,3} is an equivalence class. The pairs (4,4), (4,5), (5,4) and $(5,5) \in R \Leftarrow A_2$ = {4,5} is an equivalence class.
The pair $(6,6) \in R \Leftarrow$ {6} is an equivalence class.

No pairs other than those listed belongs to R.