Chapter 4: Counting

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Basics of Counting (4.1)

- Introduction
  - Study of gambling games
  - Complexity of algorithms
  - Probabilities of events
  - Allowable passwords on a computer system
  - Pigeonhole problem (among a set of 15 or more students at least 3 were born on the same day of the week)

Basics of Counting (4.1) (cont.)

- Basics counting principles

  - Product rule

Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are $n_1 n_2$ ways to do the procedure.

Basics of Counting (4.1) (cont.)

- Example: There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?

  Solution: The procedure of choosing consists of two tasks:

  1. Picking a microcomputer
  2. Picking a port on this microcomputer

  Since there are 32 ways to choose the microcomputer and 24 to choose the port no matter which microcomputer has been selected.

  Product rule $\Rightarrow$ 32 * 24 = 768 ports.

Basics of Counting (4.1) (cont.)

- Product rule is often phrased in terms of sets:

  If A_1 , A_2, …, A_m are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements in each set

$$|A_1 * A_2 * \ldots * A_m| = |A_1| * |A_2| * \ldots * |A_m|$$

Basics of Counting (4.1) (cont.)

- The sum rule

  If a first task can be done in n_1 ways and a second task in n_2 ways, and if theses tasks cannot be done at the same time, then there are $n_1 + n_2$ ways to do one of these tasks

Basics of Counting (4.1) (cont.)

- Example: What is the value of k after the following code has been executed?

```
k := 0
for i1 := 1 to n1
      k := k + 1
for i2 := 1 to n2
      k := k + 1
            …
for im := 1 to nm
      k := k + 1
```

Basics of Counting (4.1) (cont.)

Solution:

Initial value of k = 0

m = different loops

Each time a loop is traversed → k+1

T_i = task of traversing the ith loop.

The task T_i can be done in n_i ways, since the ith loop is traversed n_i times.

Since no two of these tasks can be done at the same time, the sum rule shows that the final value of k, which is the number of ways to do one of the tasks T_i, i = 1, 2, …, m, is $n_1 + n_2 + \ldots + n_m$.

Basics of Counting (4.1) (cont.)

- The sum rule can be phrased in terms of sets

  If $A_1, A_2, \ldots, A_m$ are disjoint sets then the number of elements in the union of theses sets is the sum of the numbers of the elements in them.

$$|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \ldots + |A_m|$$

Basics of Counting (4.1) (cont.)

- The inclusion –exclusion principle

  - When two tasks can be done at the same time, we cannot use the sum rule to count the number of ways to do one of the two tasks.

  - To correctly count the number of ways to do one of the two tasks, we add the number of ways to do each of the two tasks and then subtract the number of ways to do both tasks. This technique is called the principle of inclusion-exclusion

Basics of Counting (4.1) (cont.)

- Example: How many bit strings of length eight either start with a 1 bit or end with a the two bits 00?

Solution:

- Number of bit strings of length 8 beginning with a 1 bit = 2^7 = 128 ways (first bit can be chosen in only one way; and each of the other seven bits can be chosen in two ways $\Rightarrow$ product rule: $2^7 * 1 = 2^7$)
- Number of bits strings of length 8 ending with the 2 bits 00 = 2^6 = 64 ways (each of the first six bits can be chosen in two ways and the last two bits can be chosen in only one way) $\Rightarrow$ product rule: $2^6 * 1 = 2^6$

Basics of Counting (4.1) (cont.)

Solution (cont.):

- Both previous tasks constructs a bit string of length 8 that begins with 1 and ends with 00 which are 2^5 = 32 ways (first bit can be chosen in only one way, each of the second through the sixth bits can be chosen in two ways, and the last two bits can be chosen in only one way $\Rightarrow 2^5 * 1 * 1 = 2^5$
- Conclusion: the number of bit strings of length 8 that begin with a 1 bit or end with a 00 = 128 + 64 – 32 = 160.

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

The Pigeonhole Principle (4.2)

- Introduction
  - Suppose that a flock of pigeons flies into a set of pigeonholes to roost. The pigeonhole principle states that there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it
  - Pigeonhole principle theorem

If k + 1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects

The Pigeonhole Principle (4.2) (cont.)

Proof: Assume that none of the k boxes contains more than one object. Then the total number of objects would be at most k. This is a contradiction, since there are at least k + 1 objects.

- The pigeonhole principle is also known as the Dirichlet drawer principle (French mathematician of the 19^{th} century)

The Pigeonhole Principle (4.2) (cont.)

- Examples:
  - Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays
  - In any group of 27 English words, there must be at least two that begin with the same letter, since there are 26 letters in the English alphabet

The Pigeonhole Principle (4.2) (cont.)

- The generalized pigeonhole principle
  - The number of objects exceeds a multiple of the number of boxes
  - When 21 objects are distributed into 10 boxes. One box must have more than 2 objects

The Pigeonhole Principle (4.2) (cont.)

- The generalized pigeonhole principle theorem

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects

Proof: Suppose that none of the boxes contains more than $\lceil N/k \rceil$ - 1 objects. Then, the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

where the inequality $\lceil N/k \rceil < (N/k) + 1$ has been used. This is a contradiction since there are a total of N objects.

The Pigeonhole Principle (4.2) (cont.)

- Example: Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month
- Example: What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D and F?

Solution: The minimum number of students needed to ensure that at least 6 students receive the same grade is the smallest integer N such that $\lceil N/5 \rceil = 6$. The smallest such integer is
N = 5 * 5 + 1 = 26. If you have only 25 students, it is possible for there to be five who have received each grade so that no 6 students have received the same grade. Thus, 26 is the minimum number of students needed to ensure that at least 6 students will receive the same grade

Permutations & Combinations (4.3)

- Introduction

  - A tennis team has ten members
  - The coach has to select 5 players to make the trip to a match at another school
  - In addition, the coach has to prepare an ordered list of 4 players to play the 4 single matches

  Our goal in this section is to count the different unordered collections of the 5 players to make the trip; and the different ordered lists of 4 players to play single matches.

Permutations & Combinations (4.3) (cont.)

- Permutations

  - A permutation of a set of distinct objects is an ordered arrangements of these objects. An ordered arrangement of r elements of a set is called an r-permutation

  - Example: Let S = {1, 2, 3, }. The arrangement 3, 1, 2 is a permutation of S. The arrangement 3, 2, is a 2-permutation of S

Permutations & Combinations (4.3) (cont.)

- Theorem 1:

> The number of r-permutations of a set with n distinct element is:
>
> $$P(n, r) = n(n-1)\,(n-2)\,\ldots\,(n-r+1).$$

Proof:

- First element can be chosen in n ways
- Second element can be chosen in $(n-1)$ ways

  …
- rth element can be chosen in $(n-r+1)$ ways

  Finally using the product rule, there are
  $n(n-1)(n-2)\ldots(n-r+1)$ ways $\Rightarrow$ r-permutations of the set.
  Consequence: $P(n, r) = n\,(n-1)\,(n-2)\,\ldots\,(n-r+1) = n! / [(n-r)!]$

  $P(n, n) = n!$

Permutations & Combinations (4.3) (cont.)

- Example: How many ways are there to select a first-prize winner, a second-prize winner and a third-prize winner from 100 different people who have entered a contest?

  Solution: Because it matters which person wins which prize, the number of ways to pick the 3 prize winners is the number of ordered selections of three elements from a set of 100 elements, that is, the number of 3 permutations of a set of 100 elements.

  $$P(100, 3) = 100 * 99 * 98 = 970,200$$

Permutations & Combinations (4.3) (cont.)

- Combinations
  - An r-combination of elements of a set is an <u>unordered</u> selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.
  - Example: Let S be the set {1, 2, 3, 4}. Then {1, 3, 4} is a 3-combination from S.
  - Number of r-combinations of a set with n distinct elements is denoted by C(n, r)
  - Example: We see that C(4,2) = 6, since the two-combinations of {a, b, c, d} are the six subsets {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}

Permutations & Combinations (4.3) (cont.)

- Theorem 2:

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$, equals

$$C(n,r) = \frac{n!}{r!(n-r)!}.$$

Permutations & Combinations (4.3) (cont.)

- Corollary 1

Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Proof: From theorem 2 it follows that $C(n,r) = \frac{n!}{r!(n-r)!}$.

and $C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!\,r!}$.

Hence, $C(n, r) = C(n, n - r)$.

Permutations & Combinations (4.3) (cont.)

- Definition 1:

A combinatory proof is a proof that uses counting arguments to prove a theorem, rather than some other method such as algebraic techniques.

Many identities involving binomial coefficients can be proved using combinatorial proofs. An identity can be proved using a combinatorial proof if it can be shown that the two sides of the identity count the same elements, but in different ways.

Permutations & Combinations (4.3) (cont.)

- Example: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?

  Solution: The answer is given by the number of 5-combinations of a set with 10 elements. By theorem 2, the number of such combinations is

$$C(10, 5) = 10! / (5!\ 5!) = 252.$$

Permutations & Combinations (4.3) (cont.)

- Example: How many bit strings of length n contain exactly r 1s?

  Solution: The positions of r 1s in a bit string of length n form an r-combination of the set {1, 2, 3, …, n}. Hence, there are C(n, r) bit strings of length n that contain exactly r 1s.

Generating Permutations & Combinations (4.6)

- We are capable to count how many permutation we can obtain to solve a problem
- However, we were unable to sequentially generate one by one in order to test a certain property of a permutation
- Examples:
  - Salesman problem: a salesman must visit 6 different cities. In which order must these cities be visited in order to minimize the total travel time?
    (determine travel time one by one among the 6! = 720 possibilities!)
  - Some numbers from a set of six numbers have 100 as their sum. One way to find these numbers is to generate all $2^6 = 64$ subsets and check the sum of their terms

Generating Permutations & Combinations (cont.)

- Generating permutations
  - Any set with n elements can be placed in a one-to-one correspondence with the set {1, 2, …, n}
  - We can list the permutations of any set of n elements by generating the permutations of the n smallest positive integers and then replacing these integers with the corresponding elements
  - Many algorithms have been developed to generate the n! permutations of this set

Generating Permutations & Combinations (cont.)

- In the lexicographic ordering of the set of permutations of {1, 2, …, n}, the permutation $a_1a_2\ldots a_n$ precedes the permutation $b_1b_2\ldots b_n$, for some k, $1 \le k \le n$, $a_1 = b_1$; $a_2 = b_2$ …; $a_{k-1} = b_{k-1}$ and $a_k < b_k$

- Example: The permutation 23415 of the set {1, 2, 3, 4, 5}precedes the permutation 23514, since these permutations agree in the first two positions, but the number in the third position in the first permutation, 4, is smaller than the number in the third position in the second permutation, 5. Similarly, the permutation 41532 precedes 52143.

- Generating the next largest permutation

a) Case $a_{n-1} < a_n$: interchange a_{n-1} and a_n to obtain a larger permutation
Example: Next (234156) = 234165

b) Case $a_{n-1} > a_n \Rightarrow$ a larger permutation cannot be obtained by interchanging these last two terms

c) Case $a_{n-2} > a_{n-1}$
- ➢ Look at the last 3 integers in the permutation. If $a_{n-2} < a_{n-1} \Rightarrow$ rearrange the last 3 integers to obtain a next larger permutation
- ➢ Put the smaller of the two integers a_{n-1} and a_n that is greater than a_{n-2} in position n –2
- ➢ Place the remaining integer and a_{n-2} into the last two positions in increasing order

- Example: Next (234165) = 234516

- Algorithm: Generating the next largest permutation ion lexicographic order

```
procedure next permutation (a1a2… an: permutation of
     {1,2,…,n} not equal to n n-1 … 2 1)
j := n - 1
While aj > aj+1
     j := j - 1
{j is the largest subscript with aj < aj+1}
k := n
While aj > ak
     k := k - 1
{ak is the smallest integer greater than aj to the right of aj}
r := n
s := j + 1
while r > s
begin
     interchange ar and as
     r := r - 1
     s := s +1
end
{this puts the tail end of the permutation after the jth position in
increasing order}
```

Generating Permutations & Combinations (cont.)

- Generating Combinations
  - Since a combination is a subset, we can use the correspondence between subsets of $\{a_1, a_2, \ldots, a_n\}$ and bit strings of length n

    $1 \Rightarrow a_k$ is in the subset
    $0 \Rightarrow a_k$ is not in the subset
  - $\{a, b, c, d\}$ is a set of cardinality = 4
    cda is a subset $\Leftrightarrow$ 1011 (string of length 4)
  - We need to generate the next largest bit string

Generating Permutations & Combinations (cont.)

- Strategy for string of length n
  - Start with the bit string 000…00 with n zeros
  - Successively find the next largest expansion until the bit string 111…11 is obtained
  - At each stage the next largest binary expansion is found by locating the first position from the right that is not 1
  - Change all the 1s to the right of this position to 0s and making this first 0 (from the right) a 1

Generating Permutations & Combinations (cont.)

- Example: Find the next largest bit string after 10 0010 0111.

  Solution:
  the first bit from the right that is not a 1 is the 4^{th} bit from the right. Change this bit to a 1 and change all the following bits to 0s. This produces the next largest bit string, 10 0010 1000

- Algorithm: Generating the next r-combination in lexicographic order

procedure next r-combination ($\{a_1, a_2, \ldots, a_r\}$: proper subset of $\{1, 2, \ldots, n\}$ not equal to $\{n - r + 1, \ldots, n\}$ with $a_1 < a_2 < \ldots < a_r$)

```
i := r
While a_i = n − r + i
     i := i − 1
a_i := a_i + 1
For j := i + 1 to r
  a_j := a_i + j - i
```