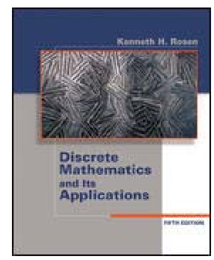
Ch.1 (Part 4): The Foundations: Logic and Proof, Sets, and Functions

- Functions (Section 1.8)



Functions (1.8)

- Definition:

  Let A and B be sets. A function (mapping, map) f from A to B, denoted f :A→B, is a subset of A*B such that

$$\forall x\, [x \in A \rightarrow \exists y\, [y \in B \wedge \langle x, y \rangle \in f\,]]$$

and

$$[\langle x, y_1 \rangle \in f \wedge \langle x, y_2 \rangle \in f\,] \rightarrow y_1 = y_2$$

Functions (1.8) (cont.)

- **Note:** f associates with each x in A one and only one y in B.
  A is called the *domain* and
  B is called the *codomain*.
  If f(x) = y
  y is called the *image* of x under f
  x is called a *preimage* of y

  (note there may be more than one preimage of y but there is only one image of x).

  The *range* of f is the set of all images of points in A under f. We denote it by f(A).

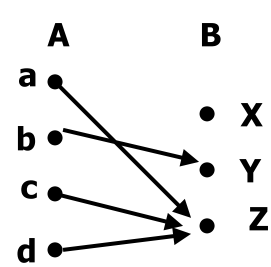
Functions (1.8) (cont.)

If S is a subset of A then

f(S) = {f(s) | s in S}.

Example:

- f(a) = Z
- the image of d is Z
- the domain of f is A = {a, b, c, d}
- the codomain is B = {X, Y, Z}
- f(A) = {Y, Z}
- the preimage of Y is b
- the preimages of Z are a, c and d
- f({c,d}) = {Z}



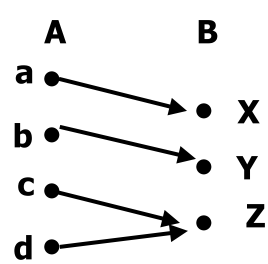
Functions (1.8) (cont.)

- Injections, Surjections and Bijections
  - Let f be a function from A to B.
  - **Definition:** f is *one-to-one* (denoted 1-1) or *injective* if preimages are unique.
    Note: this means that if a ≠ b then f(a) ≠ f(b).
  - **Definition:** f is *onto* or *surjective* if every y in B has a preimage.
    Note: this means that for every y in B there must be an x in A such that f(x) = y.
  - **Definition:** f is *bijective* if it is surjective and injective (one-to-one and onto).

Functions (1.8) (cont.)

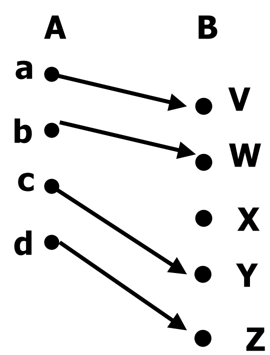
- **Examples:**

The previous Example function is neither an injection nor a surjection. Hence it is not a bijection.

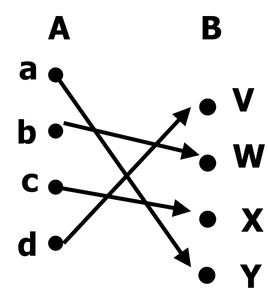


Surjection but not an injection

Functions (1.8) (cont.)



Injection but not a surjection



Injection & a surjection, hence a bijection

Functions (1.8) (cont.)

- Note: Whenever there is a bijection from A to B, the two sets must have the same number of elements or the same *cardinality*.
- That will become our *definition*, especially for infinite sets.

Functions (1.8) (cont.)

- **Examples:**

  Let A = B = R, the reals. Determine which are injections, surjections, bijections:

  - $f(x) = x$,
  - $f(x) = x^2$,
  - $f(x) = x^3$,
  - $f(x) = x + \sin(x)$,
  - $f(x) = | x |$

Functions (1.8) (cont.)

- Let E be the set of even integers {0, 2, 4, 6, ...}.

  Then there is a bijection f from *N* to *E*, the even nonnegative integers, defined by

  $$f(x) = 2x.$$

  Hence, the set of even integers has the <u>same</u> cardinality as the set of natural numbers.

  OH, NO! IT CAN'T BE....*E* IS ONLY HALF AS BIG!!!

  Sorry! It gets worse before it gets better.

Functions (1.8) (cont.)

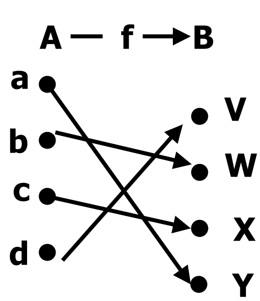
- Inverse Functions
  - **Definition:**

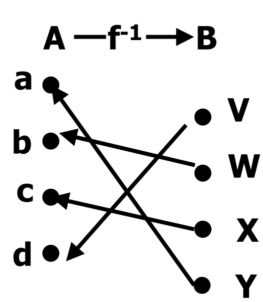
    Let f be a bijection from A to B. Then the *inverse* of f, denoted f^{-1}, is the function from B to A defined as

    $$f^{-1}(y) = x \text{ iff } f(x) = y$$

Functions (1.8) (cont.)

- **Example:** Let f be defined by the diagram:





Note: No inverse exists unless f is a bijection

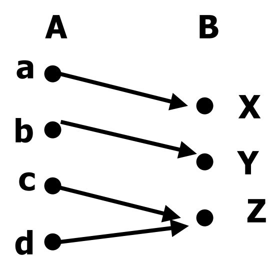
Functions (1.8) (cont.)

- **Definition:** Let S be a subset of B. Then

  $f\text{-}1(S) = \{x \mid f(x) \in S\}$

  Note: f need not be a bijection for this definition to hold.

- **Example:** Let f be the following function:



$f^{-1}(\{Z\}) = \{c, d\}$
$f^{-1}(\{X, Y\}) = \{a, b\}$

Functions (1.8) (cont.)

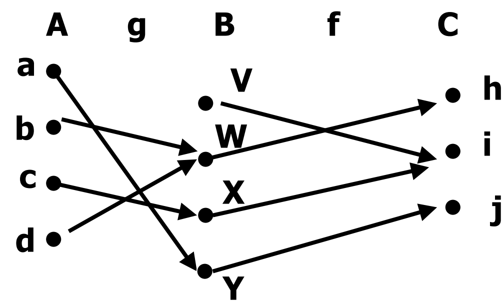
- Composition

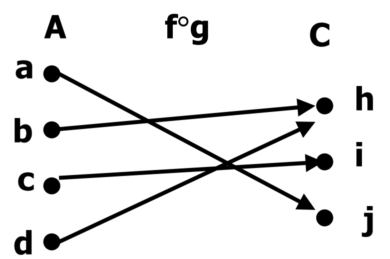
  - **Definition:**

    Let f: B→ C, g: A→ B. The *composition of f with g*, denoted f°g, is the function from A to C defined by

    $$f \circ g(x) = f(g(x))$$

- **Examples:**





Functions (1.8) (cont.)

- If $f(x) = x^2$ and $g(x) = 2x + 1$, then $f(g(x)) = (2x+1)^2$ and $g(f(x)) = 2x^2 + 1$

- **Definition:**
  - The *floor* function, denoted $f(x) = \lfloor x \rfloor$ or $f(x) = \text{floor}(x)$, is the largest integer less than or equal to x.
  - The *ceiling* function, denoted $f(x) = \lceil x \rceil$ or $f(x) = \text{ceiling}(x)$, is the smallest integer greater than or equal to x.

Functions (1.8) (cont.)

- **Examples:** $\lfloor 3.5 \rfloor = 3$, $\lceil 3.5 \rceil = 4$.
  Note: the floor function is equivalent to truncation for positive numbers.

- **Example:**
  Suppose f: B → C, g: A → B and f ∘ g is injective.
  What can we say about f and g?

  - We know that if $a \neq b$ then $f(g(a)) \neq f(g(b))$ since the composition is injective.
  - Since f is a function, it cannot be the case that $g(a) = g(b)$ since then f would have two different images for the same point.
  - Hence, $g(a) \neq g(b)$

  It follows that g must be an injection.
  However, f need not be an injection (you show).